How to Verify Quantum Processes

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Plan

- Electronics as the standard support for computers

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A Brief History of Quantum Computing

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- 1994, Shor: quantum algorithm for prime factorisation ($O(\log^3(n))$)

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E.g. $H := \frac{1}{\sqrt{2}} \begin{array}{cc} & \begin{array}{cc} |0\rangle & |1\rangle \end{array} \\ \begin{array}{c} |0\rangle \\ |1\rangle \end{array} & \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \end{array}$ "quantum coin toss"

Quantum Is More Than Probabilistic

$$\frac{|0\rangle+|1\rangle}{\sqrt{2}}$$

$$\begin{cases} |0\rangle & 50\% \\ |1\rangle & 50\% \end{cases}$$

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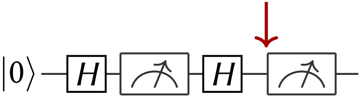

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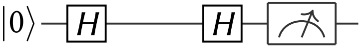

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- An entangled state cannot be broken down as $q_0 \otimes q_1$

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$$= \frac{1}{\sqrt{2}}\,|0\rangle\!\langle 0| + \frac{1}{\sqrt{2}}\,|0\rangle\!\langle 1| + \frac{1}{\sqrt{2}}\,|1\rangle\!\langle 0| - \frac{1}{\sqrt{2}}\,|1\rangle\!\langle 1|$$

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$$H\,|0\rangle = \tfrac{1}{\sqrt{2}}\left[|0\rangle\left\langle 0\middle|0\right\rangle + |0\rangle\left\langle 1\middle|0\right\rangle + |1\rangle\left\langle 0\middle|0\right\rangle - |1\rangle\left\langle 1\middle|0\right\rangle\right] = \tfrac{|0\rangle+|1\rangle}{\sqrt{2}}$$

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- $|+\rangle := H\,|0\rangle = \frac{|0\rangle+|1\rangle}{\sqrt{2}}$ and $|-\rangle := H\,|1\rangle = \frac{|0\rangle-|1\rangle}{\sqrt{2}}$ (($|0\rangle$, $|1\rangle$) and ($|+\rangle$, $|-\rangle$) are bases of $\mathbb{C}^2$)

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- $\mathrm{QFT}_2 \circ \overbrace{|0+\rangle}^{\substack{\text{state preparation} \\ =|0\rangle\otimes|+\rangle}} = \frac{1}{2}\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & i & -1 & -i \\ 1 & -1 & 1 & -1 \\ 1 & -i & -1 & i \end{pmatrix} \circ \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix} = \frac{1}{2\sqrt{2}}\begin{pmatrix} 2 \\ 1+i \\ 0 \\ 1-i \end{pmatrix}\begin{array}{c} |00\rangle \\ |01\rangle \\ |10\rangle \\ |11\rangle \end{array}$

measurement $\to$ 50% $|00\rangle$, 25% $|01\rangle$, 25% $|11\rangle$

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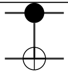
Quantum gates:

X or *Not*

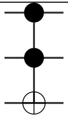
$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

CX or *CNot*



$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} = |a, b\rangle \mapsto |a, a \oplus b\rangle$$

Toffoli or *CCX* or *CCNot*



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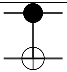

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

$$= |a, b\rangle \mapsto |a, a \oplus b\rangle$$

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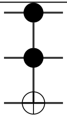

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H

$$-\boxed{H}-$$

$$\frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

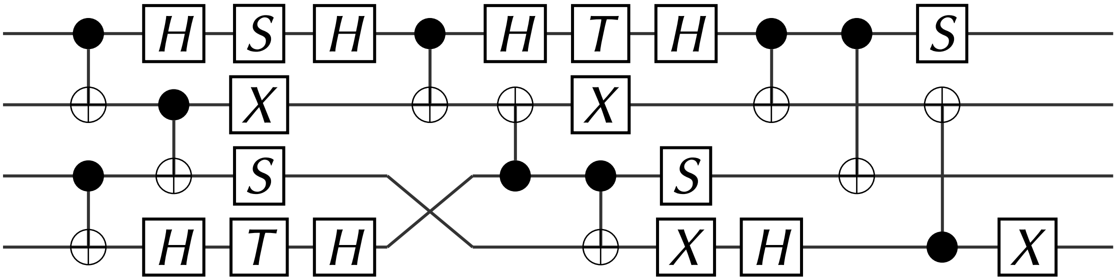
$Z(\alpha)$ or $R_Z(\alpha)$

$$-\boxed{Z(\alpha)}-$$

$$\begin{pmatrix} 1 & 0 \\ 0 & e^{i\alpha} \end{pmatrix}$$

$$-\boxed{S}- := -\boxed{Z(\tfrac{\pi}{2})}-$$

$$-\boxed{T}- := -\boxed{Z(\tfrac{\pi}{4})}-$$


H
S
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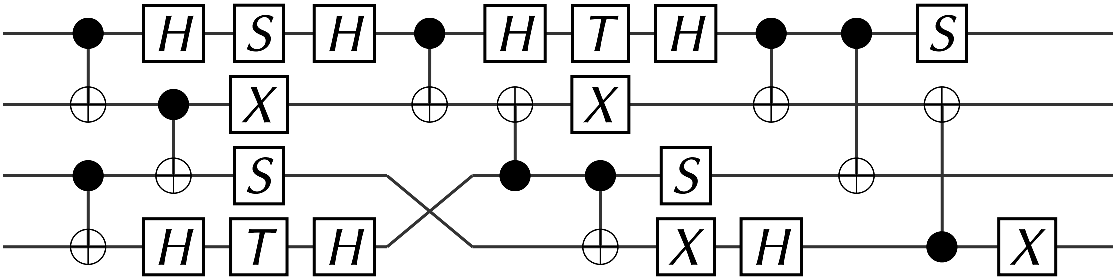
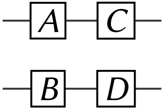
H
X



NB: make the equation $(C \otimes D)(A \otimes B) = CA \otimes DB$ obvious:



- Qubit initialisation:

Init

$$\boxed{\begin{array}{c} |0\rangle\!-\!\!\!-\!\!\!- \\ \begin{pmatrix}1\\0\end{pmatrix} \end{array}}$$

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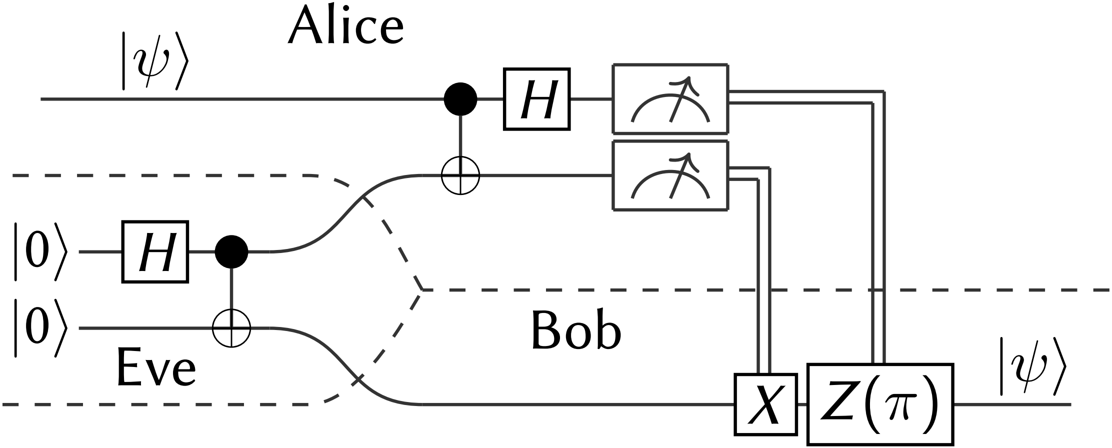
- Measurement (effect + classical information):

Meas

$$\left\{\left\langle\begin{pmatrix}1 & 0\end{pmatrix},0\right\rangle,\left\langle\begin{pmatrix}0 & 1\end{pmatrix},1\right\rangle\right\}$$


Alice
$|\psi\rangle$
H
$|0\rangle$
$|0\rangle$
H
Eve
Bob
X
$Z(\pi)$
$|\psi\rangle$

Theorem : Universality[1]

The gate set $\{H, Z(\alpha), CX\}_{\alpha\in\mathbb{R}}$ is universal.

[1] [Barenco *et al.*'95]
[2] Gottesman-Knill theorem, [Gottesman'98]
[3] [Boykin, Mor, Pulver, Roychowdhury, Vatan' 00]
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  - not universal
  - efficiently simulable on a classical computer[2]

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- Clifford+T fragment : $\alpha \in \frac{\pi}{4}\mathbb{Z}$
  - approx. universal[3], with efficient approximation[4]

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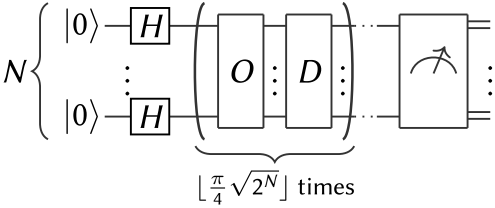

$$O : |\vec{x}\rangle \mapsto (-1)^{\delta_{A[\vec{x}],\vec{y}}} |\vec{x}\rangle$$

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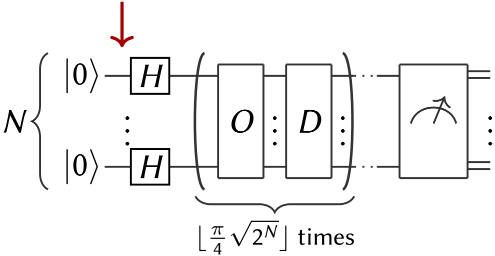

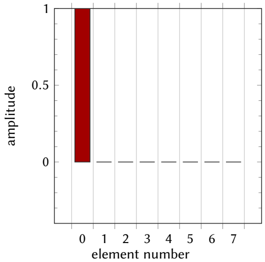

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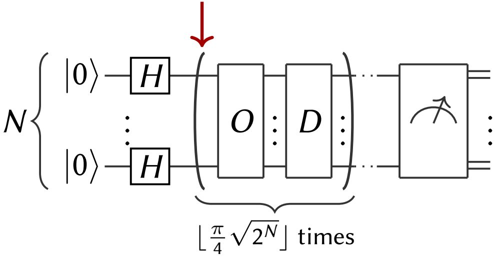

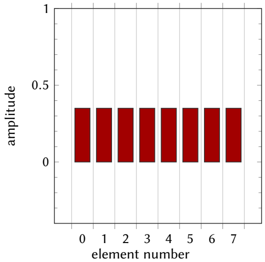

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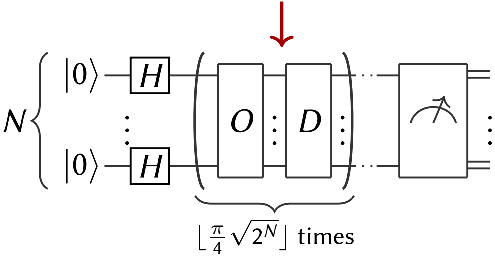

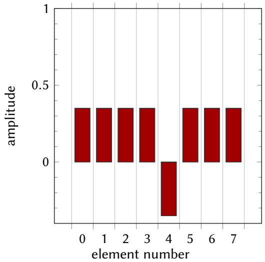

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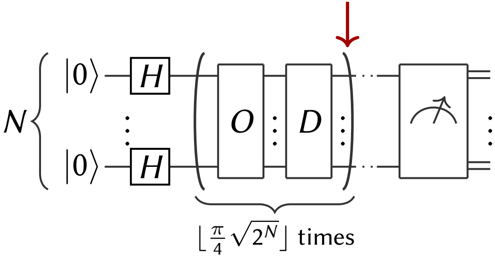

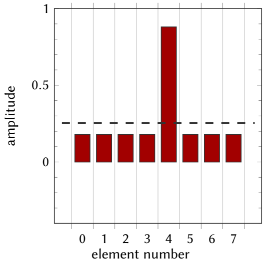

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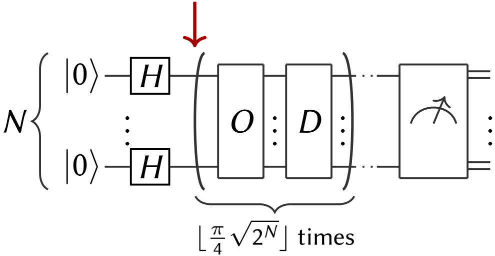

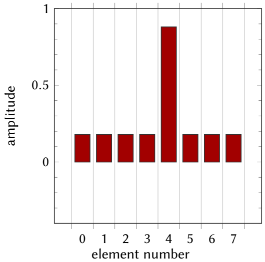

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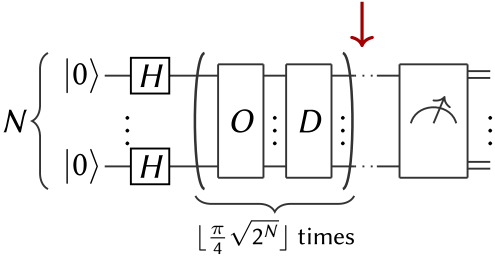

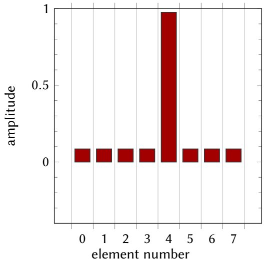

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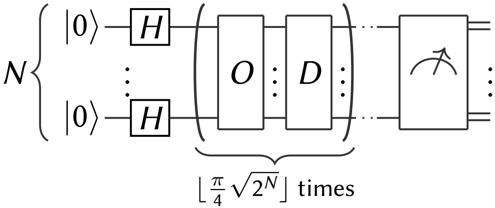

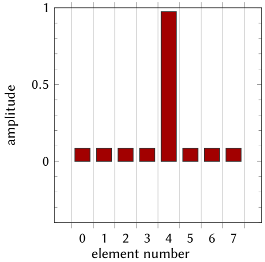

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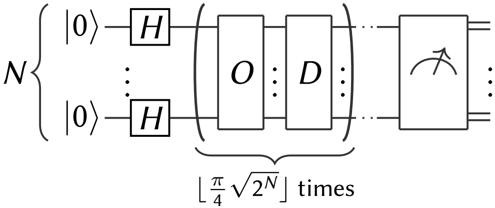

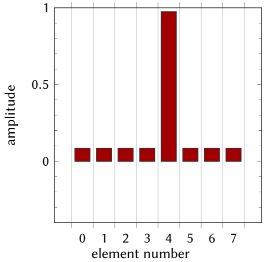

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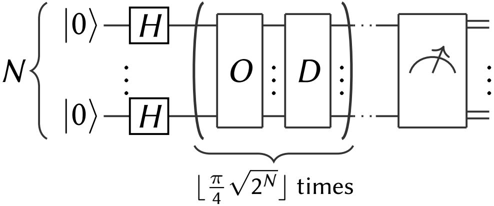

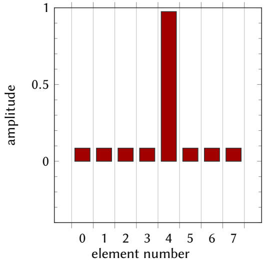

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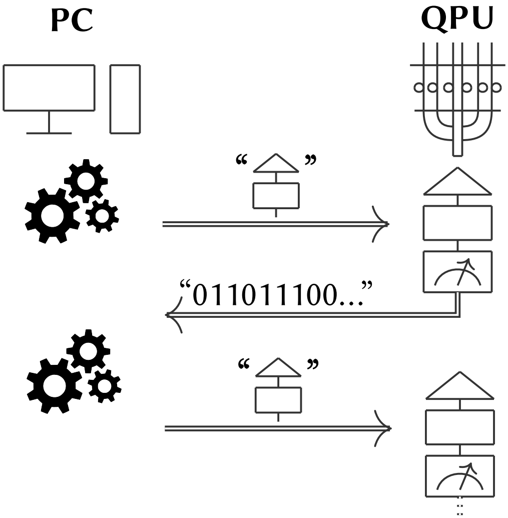
Algo in $O(\sqrt{2^N})$ vs. $O(2^N)$ classically

- Quantum counting (quadratic speedup)
- Existence of Hamiltonian cycle (quadratic speedup)
- Shor's prime factorisation (exponential speedup)
- HHL's solution to s-sparse (Hermitian) system of equations (exponential speedup)
- QAOA for combinatorial optimisation
- VQE for ground eigenvalue estimation
- ...


PC
QPU
“011011100...”

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But:

$$\left|\psi\right\rangle \otimes \left|\psi\right\rangle = \alpha^2\left|00\right\rangle + \alpha\beta(\left|01\right\rangle + \left|10\right\rangle) + \beta^2\left|11\right\rangle$$

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Quantum Lambda Calculus (Lineal [Arrighi,Dowek'06])

- $\mathbf{t} ::= \quad \mathbf{x} \quad | \quad \lambda \mathbf{x}\, \mathbf{t} \quad | \quad \mathbf{t}\ \mathbf{t} \quad | \quad \mathbf{0} \quad | \quad \alpha.\mathbf{t} \quad | \quad \mathbf{t} + \mathbf{t}$

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  E.g. $\mathbf{H} \equiv \lambda \mathbf{y}\, \{\mathbf{y}\, [\frac{\sqrt{2}}{2}.(\mathbf{false} + \mathbf{true})]\, [\frac{\sqrt{2}}{2}.(\mathbf{false} - \mathbf{true})]\}$

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Quipper ([Green,Lumsdaine,Ross,Selinger,Valiron'13])

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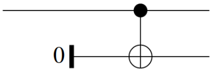

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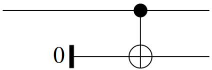

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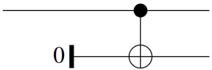

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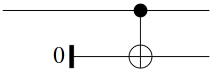

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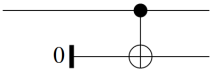

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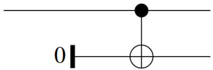

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  - Type safety
  - Reduction termination
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    - Dynamic lifting

- Qiskit
- Liqui$|\rangle$, Q#
- ProjectQ
- CirQ
- Strawberry Fields
- AQASM
- ...

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Projective Measurements for Assertions

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E.g. on 1 qubit: $\left\{ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right\}$ or $\left\{ \frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}, \frac{1}{2}\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} \right\}$

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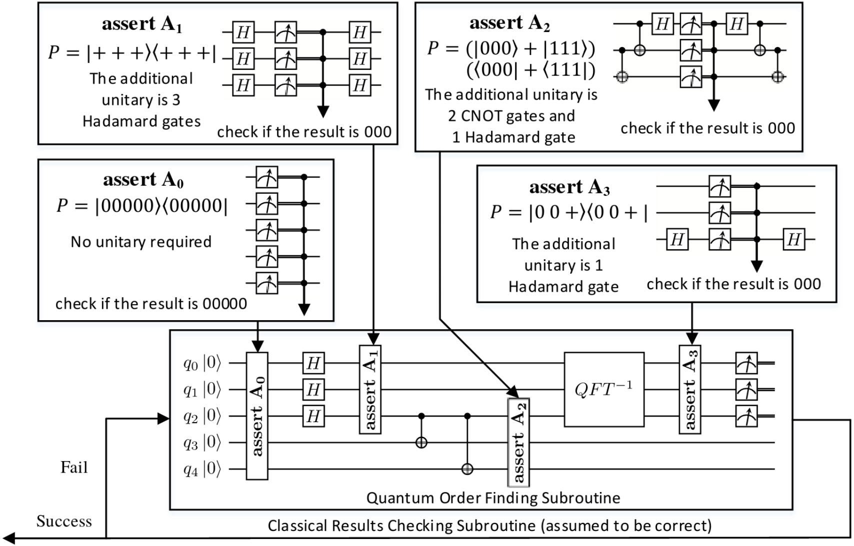
- Projection defines a subspace
  $\Rightarrow$ possible to check if state is in subspace


assert A_1
$P = |+++\rangle\langle+++|$
The additional unitary is 3 Hadamard gates
check if the result is 000
assert A_2
$P = (|000\rangle + |111\rangle)(\langle 000| + \langle 111|)$
The additional unitary is 2 CNOT gates and 1 Hadamard gate
check if the result is 000
assert A_0
$P = |00000\rangle\langle 00000|$
No unitary required
check if the result is 00000
assert A_3
$P = |0\,0\,+\rangle\langle 0\,0\,+|$
The additional unitary is 1 Hadamard gate
check if the result is 000
q_0 $|0\rangle$
q_1 $|0\rangle$
q_2 $|0\rangle$
q_3 $|0\rangle$
q_4 $|0\rangle$
QFT^{-1}
Quantum Order Finding Subroutine
Fail
Success
Classical Results Checking Subroutine (assumed to be correct)

Twist [Yuan,McNally,Carbin'22]

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- $$\begin{aligned} \text{Purity } \pi &::= \mathbf{P} \mid \mathbf{M} \\ \text{Quantum type } \varrho &::= \texttt{qubit} \mid \varrho_1 \,\&\, \varrho_2 \\ \text{Type } \tau &::= \texttt{bool} \mid \tau_1 \times \tau_2 \mid \tau_1 \to \tau_2 \mid \varrho^\pi \\ \text{Quantum value } q &::= \texttt{ref}[\alpha] \mid [q_1, q_2] \\ \text{Expression } e &::= x \mid f \mid e_1(e_2) \mid (e_1, e_2) \mid \texttt{let } (x, y) = e_1 \texttt{ in } e_2 \mid \texttt{if } e \texttt{ then } e_1 \texttt{ else } e_2 \mid \texttt{T} \mid \texttt{F} \\ &\mid \texttt{qinit } () \mid U(e) \mid U_2(e) \mid \texttt{measure}(e) \mid q^\pi \\ &\mid \texttt{entangle}_\pi(e) \mid \texttt{split}_\pi(e) \mid \texttt{cast}_\pi(e) \end{aligned}$$

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- $$\begin{aligned} \text{Purity } \pi &::= \mathbf{P} \mid \mathbf{M} \\ \text{Quantum type } \varrho &::= \mathtt{qubit} \mid \varrho_1 \,\&\, \varrho_2 \\ \text{Type } \tau &::= \mathtt{bool} \mid \tau_1 \times \tau_2 \mid \tau_1 \to \tau_2 \mid \varrho^\pi \\ \text{Quantum value } q &::= \mathtt{ref}[\alpha] \mid [q_1, q_2] \\ \text{Expression } e &::= x \mid f \mid e_1(e_2) \mid (e_1, e_2) \mid \mathtt{let}\ (x,y) = e_1\ \mathtt{in}\ e_2 \mid \mathtt{if}\ e\ \mathtt{then}\ e_1\ \mathtt{else}\ e_2 \mid \mathtt{T} \mid \mathtt{F} \\ &\mid \mathtt{qinit}\ () \mid U(e) \mid U_2(e) \mid \mathtt{measure}(e) \mid q^\pi \\ &\mid \mathtt{entangle}_\pi(e) \mid \mathtt{split}_\pi(e) \mid \mathtt{cast}_\pi(e) \end{aligned}$$
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- Type system (type safety)
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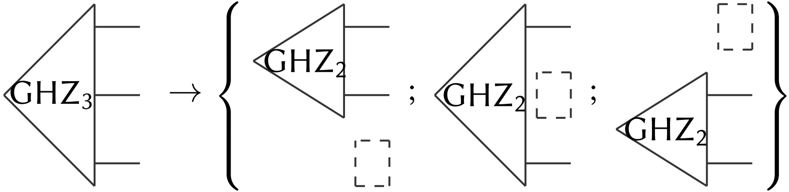
Abstract Interpretation

Define a "lossy" interpretation, much easier to compute, which still provides useful information

- [Perdrix'08]: $\mathfrak{Purity}$ check, as partition of the memory
- [Yu,Palsberg'21]: Decompose states into subspaces



- $S ::= \textbf{skip} \mid \overline{q} := |0\rangle \mid \overline{q} := U\overline{q} \mid S \mid \textbf{measure } M[\overline{q}]\overline{S} \mid \textbf{while } M[\overline{q}] \textbf{ do } \{S\}$

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- $$\{P\}\ \textbf{skip}\ \{P\}$$
  $$\{U^\dagger P U\}\ \overline{q} := U\overline{q}\ \{P\}$$
  $$\frac{\{P\}\ S_1\ \{Q\} \qquad \{Q\}\ S_2\ \{R\}}{\{P\}\ S_1; S_2\ \{R\}}$$
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  $$\frac{\{Q\}\ S\ \{M_0^\dagger P M_0 + M_1^\dagger Q M_1\}}{\{M_0^\dagger P M_0 + M_1^\dagger Q M_1\}\ \textbf{while } M[\overline{q}] = 1 \textbf{ do } S\ \{P\}}$$
  $$\frac{P \sqsubseteq P' \quad \{P'\}\ S\ \{Q'\} \quad Q' \sqsubseteq Q}{\{P\}\ S\ \{Q\}}$$

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- Löwner order: $P \sqsubseteq P' \iff P' - P$ is positive semi-definite

Example

$$
\begin{array}{l}
q := |0\rangle\ ; \\
q := Hq; \\
\textbf{while } M[q] = 1 \textbf{ do } \{ \\
\quad q := |0\rangle\ ; \\
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\}
\end{array}
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$$
\dfrac{
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\quad
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$$

$$
|0\rangle\langle 0| + |1\rangle\langle 1| - |+\rangle\langle +| = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - \frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} = |-\rangle\langle -|
$$

$$
\Rightarrow \quad |+\rangle\langle +| \sqsubseteq |0\rangle\langle 0| + |1\rangle\langle 1|
$$

- Sqir [Hietala,Rand,Hung,Li,Hicks'21]
  - Small but non-trivial syntax
  - Matrix/density operators semantics
  - Proof obligations proven using Coq
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- Voqc [Hietala,Rand,Hung,Wu,Hicks'21]
  - Verified compiler
  - Sqir as intermediate representation

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Plan

Theorem : Universality[5]

The gate set $\{H, Z(\alpha), CX\}_{\alpha\in\mathbb{R}}$ is universal.

$Z(\alpha)$ $\Rightarrow$ infinite (uncountable) family of gates
$\Rightarrow$ bad for analysis and implementability

- Clifford fragment : $\alpha \in \frac{\pi}{2}\mathbb{Z}$
  - not universal
  - efficiently simulable on a classical computer[6]
- Clifford+T fragment : $\alpha \in \frac{\pi}{4}\mathbb{Z}$
  - approx. universal[7], with efficient approximation[8]

---

[5] [Barenco *et al.*'95]
[6] Gottesman-Knill theorem, [Gottesman'98]
[7] [Boykin, Mor, Pulver, Roychowdhury, Vatan' 00]
[8] Solovay-Kitaev theorem, [Kitaev'97]

Problem of circuit equivalence

Do two given circuits implement the same operator?

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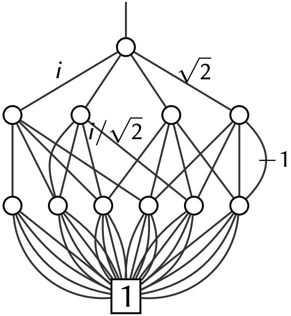
- Overall trick: try to reduce $\mathcal{C}_2^\dagger \circ \mathcal{C}_1$ to identity
- Idea 1: exploit redundancy $\Rightarrow$ Quantum-style decision diagrams
- Idea 2: reason graphically / use a rewrite system
  $\Rightarrow$ equational theory (e.g. $-\boxed{H}-\boxed{H}- \;=\; \text{———}$)

Quantum Multiple-Valued Decision Diagrams (QMDDs) [Wille et al.]

Example:



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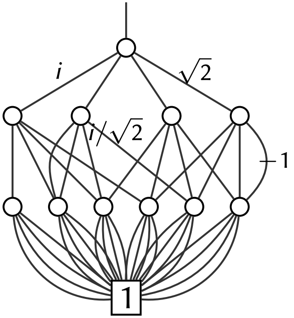
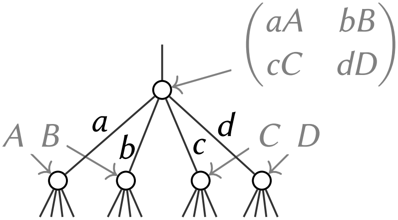


- Interpretation:



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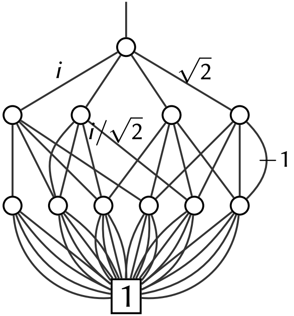

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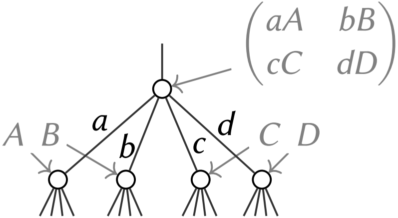

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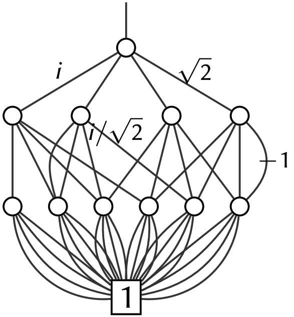

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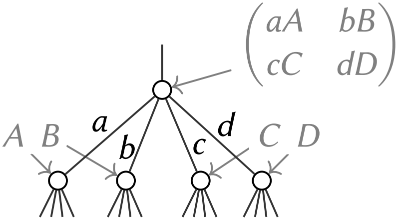

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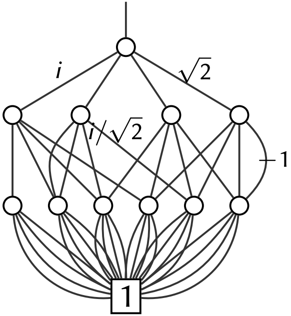

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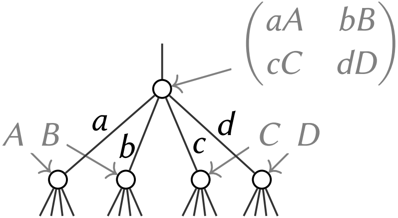

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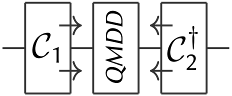

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Weak Completeness for Clifford

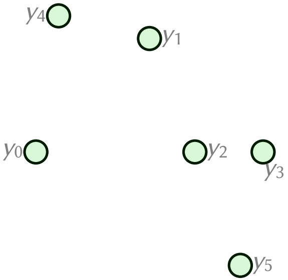
If t_0 and t_1 are unitary Clifford terms such that $[\![t_0]\!] = [\![t_1]\!]$, then $t_0 \circ t_1^\dagger \xrightarrow[\text{Clif}]{}^* id$.

$$|y_4, y_3\rangle \mapsto \sum_{\vec{y}} e^{2i\pi\left(\frac{1}{4}y_0 + \frac{1}{2}y_4y_0 + \frac{1}{8}y_5y_0y_1 + \frac{3}{4}y_1y_2y_3 + \frac{1}{2}y_0y_3\right)} |0, 1\oplus y_0 \oplus y_4y_2, y_5\rangle$$

$$\downarrow$$

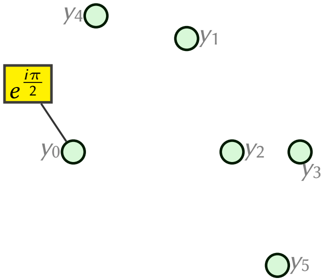
$$|y_4, y_3\rangle \mapsto \sum_{\vec{y}} e^{2i\pi\left(\frac{1}{4}y_0+\frac{1}{2}y_4y_0+\frac{1}{8}y_5y_0y_1+\frac{3}{4}y_1y_2y_3+\frac{1}{2}y_0y_3\right)} |0, 1\oplus y_0\oplus y_4y_2, y_5\rangle$$

$$\downarrow$$



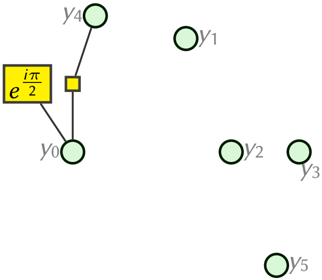
$$|y_4, y_3\rangle \mapsto \sum_{\vec{y}} e^{2i\pi\left(\frac{1}{4}y_0+\frac{1}{2}y_4y_0+\frac{1}{8}y_5y_0y_1+\frac{3}{4}y_1y_2y_3+\frac{1}{2}y_0y_3\right)}\,|0, 1\oplus y_0\oplus y_4y_2, y_5\rangle$$

$$\downarrow$$



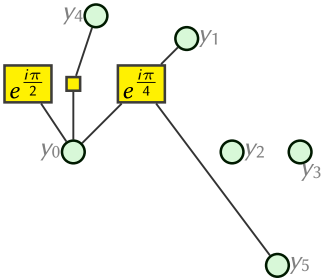
$$|y_4, y_3\rangle \mapsto \sum_{\vec{y}} e^{2i\pi\left(\frac{1}{4}y_0 + \frac{1}{2}y_4y_0 + \frac{1}{8}y_5y_0y_1 + \frac{3}{4}y_1y_2y_3 + \frac{1}{2}y_0y_3\right)} |0, 1\oplus y_0 \oplus y_4y_2, y_5\rangle$$

$\downarrow$



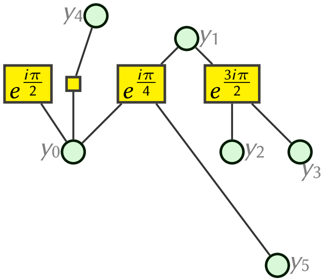
$$|y_4, y_3\rangle \mapsto \sum_{\vec{y}} e^{2i\pi\left(\frac{1}{4}y_0+\frac{1}{2}y_4y_0+\frac{1}{8}y_5y_0y_1+\frac{3}{4}y_1y_2y_3+\frac{1}{2}y_0y_3\right)}\,|0, 1\oplus y_0\oplus y_4y_2, y_5\rangle$$

$$\downarrow$$



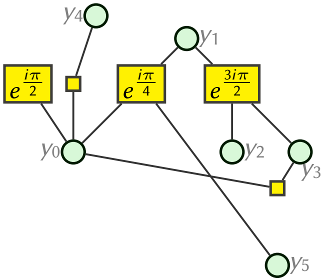
$$|y_4, y_3\rangle \mapsto \sum_{\vec{y}} e^{2i\pi\left(\frac{1}{4}y_0 + \frac{1}{2}y_4y_0 + \frac{1}{8}y_5y_0y_1 + \frac{3}{4}y_1y_2y_3 + \frac{1}{2}y_0y_3\right)} |0, 1\oplus y_0 \oplus y_4y_2, y_5\rangle$$

$$\downarrow$$



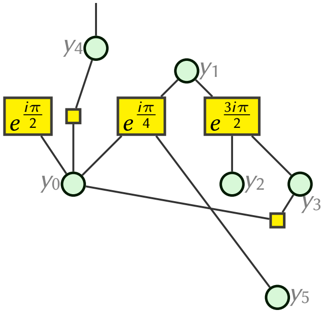
$$|y_4, y_3\rangle \mapsto \sum_{\vec{y}} e^{2i\pi\left(\frac{1}{4}y_0+\frac{1}{2}y_4y_0+\frac{1}{8}y_5y_0y_1+\frac{3}{4}y_1y_2y_3+\frac{1}{2}y_0y_3\right)}|0, 1\oplus y_0\oplus y_4y_2, y_5\rangle$$

$$\downarrow$$



$$|y_4, y_3\rangle \mapsto \sum_{\vec{y}} e^{2i\pi\left(\frac{1}{4}y_0+\frac{1}{2}y_4y_0+\frac{1}{8}y_5y_0y_1+\frac{3}{4}y_1y_2y_3+\frac{1}{2}y_0y_3\right)}|0, 1\oplus y_0\oplus y_4y_2, y_5\rangle$$

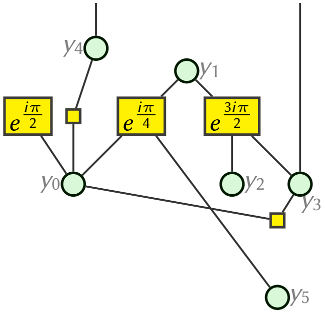
$$\downarrow$$



A Visualisation of SOP Terms With ZX Diagrams

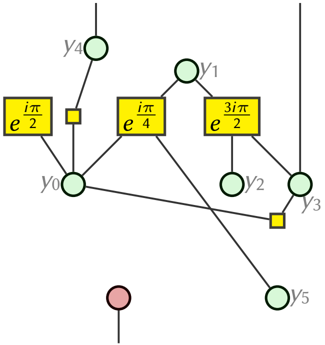
$$|y_4, y_3\rangle \mapsto \sum_{\vec{y}} e^{2i\pi\left(\frac{1}{4}y_0+\frac{1}{2}y_4y_0+\frac{1}{8}y_5y_0y_1+\frac{3}{4}y_1y_2y_3+\frac{1}{2}y_0y_3\right)} |0,\, 1\oplus y_0\oplus y_4y_2,\, y_5\rangle$$

$$\downarrow$$



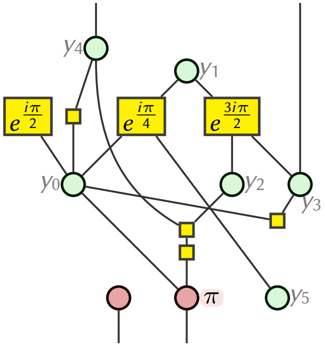
$$|y_4, y_3\rangle \mapsto \sum_{\vec{y}} e^{2i\pi\left(\frac{1}{4}y_0+\frac{1}{2}y_4y_0+\frac{1}{8}y_5y_0y_1+\frac{3}{4}y_1y_2y_3+\frac{1}{2}y_0y_3\right)}\,|0,\ 1\oplus y_0\oplus y_4y_2,\ y_5\rangle$$

$$\downarrow$$



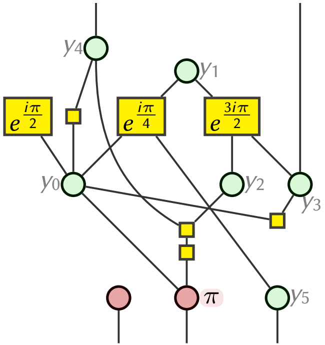
$$|y_4, y_3\rangle \mapsto \sum_{\vec{y}} e^{2i\pi\left(\frac{1}{4}y_0+\frac{1}{2}y_4y_0+\frac{1}{8}y_5y_0y_1+\frac{3}{4}y_1y_2y_3+\frac{1}{2}y_0y_3\right)} |0, 1\oplus y_0\oplus y_4y_2, y_5\rangle$$

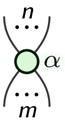
$$\downarrow$$



$$|y_4, y_3\rangle \mapsto \sum_{\vec{y}} e^{2i\pi\left(\frac{1}{4}y_0+\frac{1}{2}y_4y_0+\frac{1}{8}y_5y_0y_1+\frac{3}{4}y_1y_2y_3+\frac{1}{2}y_0y_3\right)}\,|0,\ 1\oplus y_0\oplus y_4y_2,\ y_5\rangle$$

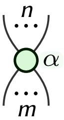
$$\downarrow$$



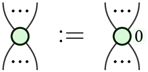
- A spider: 

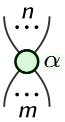
$$:: \quad 2^m \left\{ \overbrace{\begin{pmatrix} 1 & 0 & \cdots & \cdots & 0 \\ 0 & 0 & & & \vdots \\ \vdots & & \ddots & & \vdots \\ \vdots & & & 0 & 0 \\ 0 & \cdots & \cdots & 0 & e^{i\alpha} \end{pmatrix}}^{2^n} \right.$$

$$= |0^m\rangle\langle 0^n| + e^{i\alpha}|1^m\rangle\langle 1^n|$$

- A spider: 

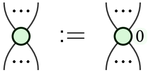
$$:: \quad 2^m \left\{ \overbrace{\begin{pmatrix} 1 & 0 & \cdots & \cdots & 0 \\ 0 & 0 & & & \vdots \\ \vdots & & \ddots & & \vdots \\ \vdots & & & 0 & 0 \\ 0 & \cdots & \cdots & 0 & e^{i\alpha} \end{pmatrix}}^{2^n} \right. = |0^m\rangle\langle 0^n| + e^{i\alpha}|1^m\rangle\langle 1^n|$$



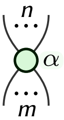
- A spider: 

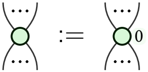
$:: \; 2^m \left\{ \overbrace{\begin{pmatrix} 1 & 0 & \cdots & \cdots & 0 \\ 0 & 0 & & & \vdots \\ \vdots & & \ddots & & \vdots \\ \vdots & & & 0 & 0 \\ 0 & \cdots & \cdots & 0 & e^{i\alpha} \end{pmatrix}}^{2^n} \right.$

$= |0^m\rangle\langle 0^n| + e^{i\alpha} |1^m\rangle\langle 1^n|$

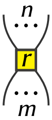


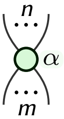
- A change of basis:  $:: \; \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$

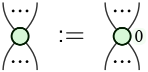
- A spider:  $:: \quad 2^m \left\{ \overbrace{\begin{pmatrix} 1 & 0 & \cdots & \cdots & 0 \\ 0 & 0 & & & \vdots \\ \vdots & & \ddots & & \vdots \\ \vdots & & & 0 & 0 \\ 0 & \cdots & \cdots & 0 & e^{i\alpha} \end{pmatrix}}^{2^n} \right. \quad = |0^m\rangle\langle 0^n| + e^{i\alpha}|1^m\rangle\langle 1^n|$

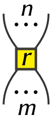


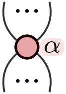
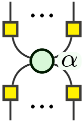
- A change of basis:  $:: \quad \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$

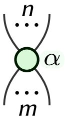
generalised to  $:: \quad \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & \cdots & \cdots & 1 \\ \vdots & \ddots & & \vdots \\ \vdots & & 1 & 1 \\ 1 & \cdots & 1 & r \end{pmatrix}$

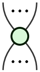
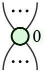
- A spider:  $:: \quad 2^m \left\{ \overbrace{\begin{pmatrix} 1 & 0 & \cdots & \cdots & 0 \\ 0 & 0 & & & \vdots \\ \vdots & & \ddots & & \vdots \\ \vdots & & & 0 & 0 \\ 0 & \cdots & \cdots & 0 & e^{i\alpha} \end{pmatrix}}^{2^n} \right.$
  $= |0^m\rangle\langle 0^n| + e^{i\alpha}|1^m\rangle\langle 1^n|$

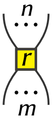


- A change of basis:  $:: \quad \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$ generalised to  $:: \quad \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & \cdots & \cdots & 1 \\ \vdots & \ddots & & \vdots \\ \vdots & & 1 & 1 \\ 1 & \cdots & 1 & r \end{pmatrix}$

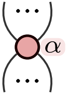
- Another spider:  $:=$  $:: \quad |+^m\rangle\langle +^n| + e^{i\alpha}\,|-^m\rangle\langle -^n|$

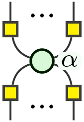
- A spider: 

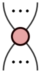
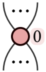
$$:: \quad 2^m \left\{ \overbrace{\begin{pmatrix} 1 & 0 & \cdots & \cdots & 0 \\ 0 & 0 & & & \vdots \\ \vdots & & \ddots & & \vdots \\ \vdots & & & 0 & 0 \\ 0 & \cdots & \cdots & 0 & e^{i\alpha} \end{pmatrix}}^{2^n} \right.$$
$$= |0^m\rangle\langle 0^n| + e^{i\alpha}|1^m\rangle\langle 1^n|$$
 $:=$ 

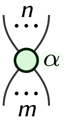
- A change of basis:  $:: \ \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$ generalised to 

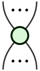
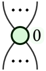
$:: \ \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & \cdots & \cdots & 1 \\ \vdots & \ddots & & \vdots \\ \vdots & & 1 & 1 \\ 1 & \cdots & 1 & r \end{pmatrix}$

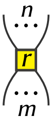
- Another spider: 

$:=$ 

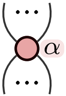
$:: \ |+^m\rangle\langle +^n| + e^{i\alpha}\,|-^m\rangle\langle -^n|$
 $:=$ 

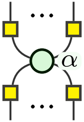
- A spider: 

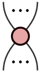
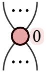
$:: \; 2^m \left\{ \overbrace{\begin{pmatrix} 1 & 0 & \cdots\cdots & 0 \\ 0 & 0 & & \vdots \\ \vdots & & \ddots & \vdots \\ \vdots & & 0 & 0 \\ 0 & \cdots\cdots & 0 & e^{i\alpha} \end{pmatrix}}^{2^n} \right. = |0^m\rangle\langle 0^n| + e^{i\alpha}|1^m\rangle\langle 1^n|$
 $:=$ 

- A change of basis:  $:: \; \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$ generalised to 

$:: \; \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & \cdots & \cdots & 1 \\ \vdots & \ddots & & \vdots \\ \vdots & & 1 & 1 \\ 1 & \cdots & 1 & r \end{pmatrix}$

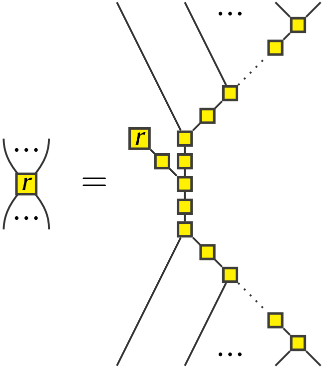
- Another spider: 

$:=$ 

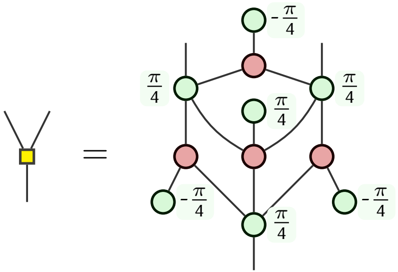
$:: \; |+^m\rangle\langle +^n| + e^{i\alpha}\,|-^m\rangle\langle -^n|$
 $:=$ 

- Wires: $:: \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$,  $:: \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$,  $:: \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix}$,  $:: \begin{pmatrix} 1 & 0 & 0 & 1 \end{pmatrix}$

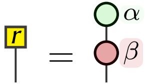
Getting Rid of the H-Spider



and

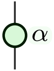


and



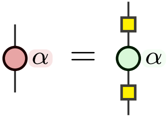
One-qubit Operators

$Z(\alpha)$:
rotation around Z



Hadamard:

$X(\alpha)$:
$= HZ(\alpha)H$



One-qubit Operators

$Z(\alpha)$:
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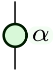
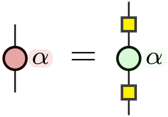


Hadamard:

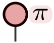
$X(\alpha)$:
$= HZ(\alpha)H$



States and Projectors

$\sqrt{2}\,|0\rangle$:

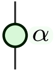
$\sqrt{2}\,|1\rangle$:



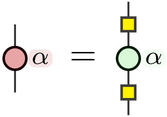
$|00\rangle + |11\rangle$:

$\langle 00| + \langle 11|$:

One-qubit Operators

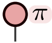
$Z(\alpha)$:
rotation around Z 

Hadamard: 

$X(\alpha)$:
$= HZ(\alpha)H$ 

States and Projectors

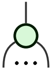
$\sqrt{2}\,|0\rangle$: 

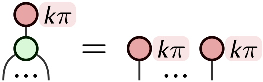
$\sqrt{2}\,|1\rangle$: 

$|00\rangle + |11\rangle$: 

$\langle 00| + \langle 11|$: 

Green Spider

Copy: 

s.t. 

One-qubit Operators

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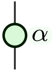
rotation around Z



Hadamard:

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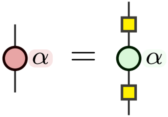

States and Projectors

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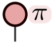
$\sqrt{2}\,|1\rangle$:

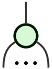


$|00\rangle + |11\rangle$:

$\langle 00| + \langle 11|$:

Green Spider

Copy:



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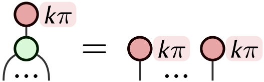
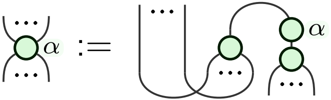
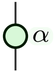




Intuition Behind the Generators

One-qubit Operators

$Z(\alpha)$:
rotation around Z



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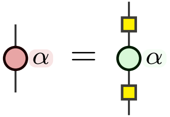

States and Projectors

$\sqrt{2}\,|0\rangle$:

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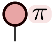
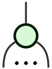


$|00\rangle + |11\rangle$:

$\langle 00| + \langle 11|$:

Green Spider

Copy:



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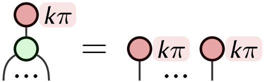
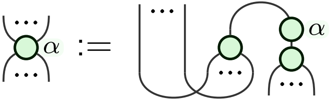
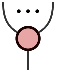




Red Spider

XOR:



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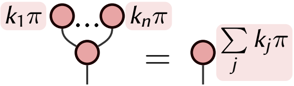

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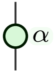

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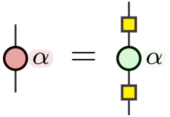

States and Projectors

$\sqrt{2}\,|0\rangle$:

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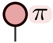

$|00\rangle + |11\rangle$:

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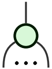
Green Spider

Copy:



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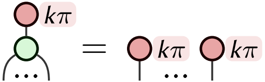
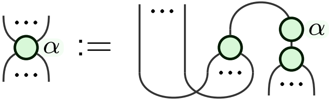
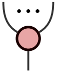




Red Spider

XOR:



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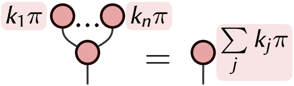
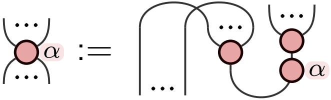
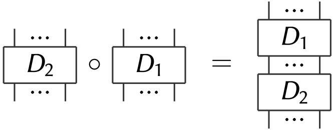


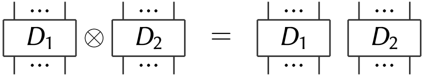


Generators

Generators

Compositions





ZX-Calculus [Coecke,Duncan'08] in Short

Generators

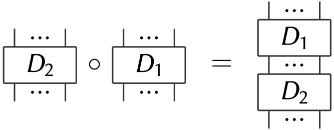
Compositions

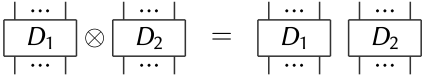
Standard Interpretation

$$[\![.]\!] : \mathbf{ZX} \to \mathcal{M}(\mathbb{C})$$

Generators

Compositions

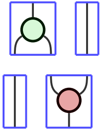




Standard Interpretation

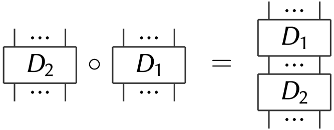
$$[\![.]\!] : \mathbf{ZX} \to \mathcal{M}(\mathbb{C})$$

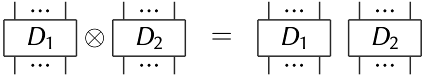
Example



Generators

Compositions





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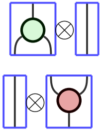
Example



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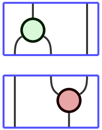

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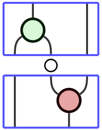

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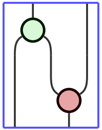
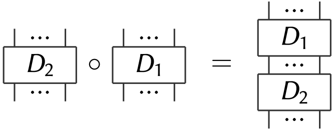
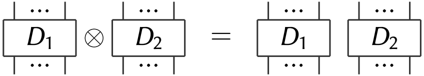


Generators

Compositions





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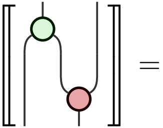
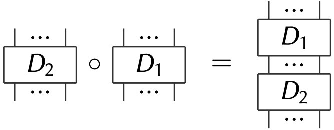
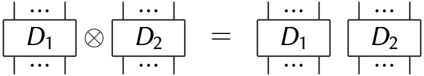
Example



Generators

Compositions





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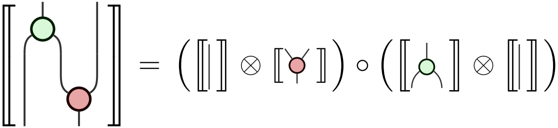
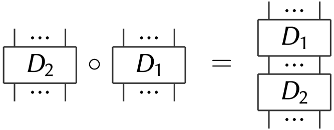
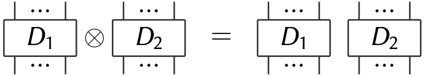
Example



Generators

Compositions





Standard Interpretation

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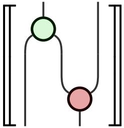
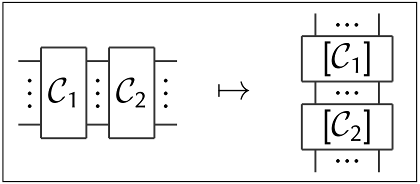
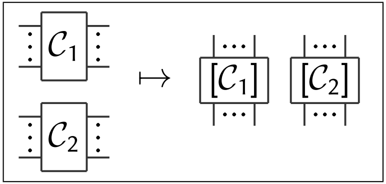
Example



$$= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$



$\mathcal{C}_1$
$\mathcal{C}_2$
$\mapsto$
$[\mathcal{C}_1]$
$[\mathcal{C}_2]$



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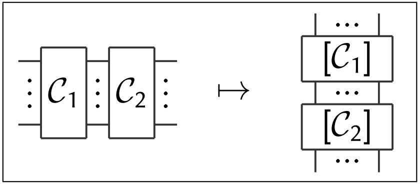
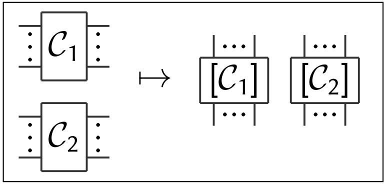
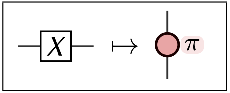


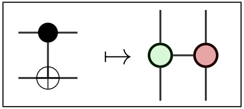
C1
C2
[C1]
[C2]

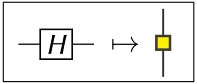


C1
C2
[C1]
[C2]

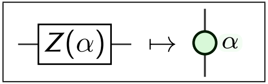


X
π

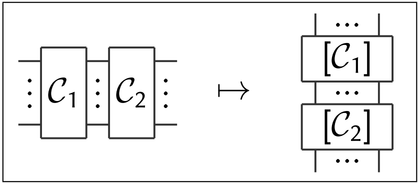


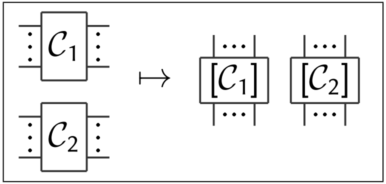


H



Z(α)
α


$\mathcal{C}_1$ $\mathcal{C}_2$ $\mapsto$ $[\mathcal{C}_1]$ $[\mathcal{C}_2]$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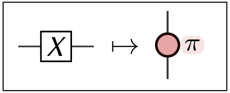
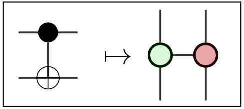
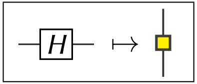
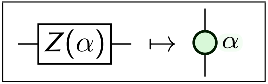
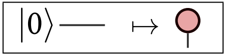
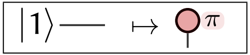

X $\mapsto$ π


$\mapsto$


H $\mapsto$


$Z(\alpha)$ $\mapsto$ α


$|0\rangle$ $\mapsto$


$|1\rangle$ $\mapsto$ π

Theorem (Universality)

We can represent any quantum operator using ZX-diagrams:

$$\forall f : \mathbb{C}^{2^n} \to \mathbb{C}^{2^m},\ \exists \overset{n}{\underset{m}{\boxed{D}}} \in \mathbf{ZX},\ \left[\!\left[\overset{n}{\underset{m}{\boxed{D}}} \right]\!\right] = f$$

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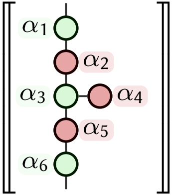
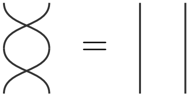
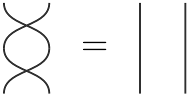
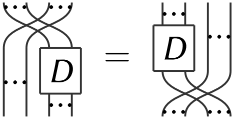
E.g. if $f : \mathbb{C}^2 \to \mathbb{C}^2$, $\exists \alpha_i$,



$= f$






D
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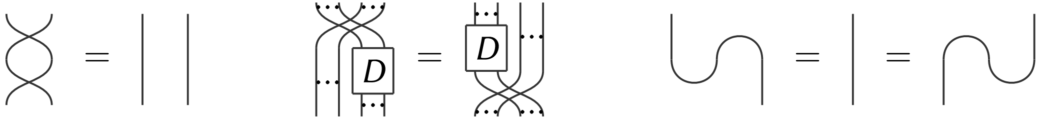
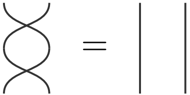
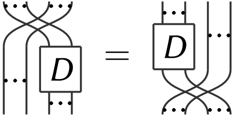
Equational Theory (the Backbone)






D
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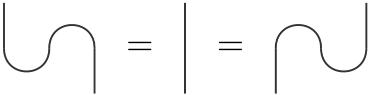
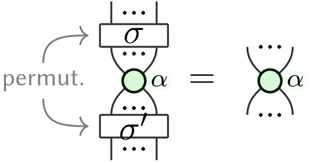
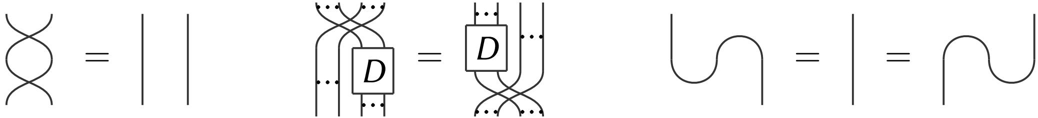



permut.
σ
α
σ'
=
α


=
D
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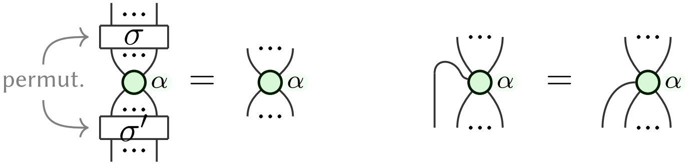
permut.
σ
α
σ′
=
α
α
=
α

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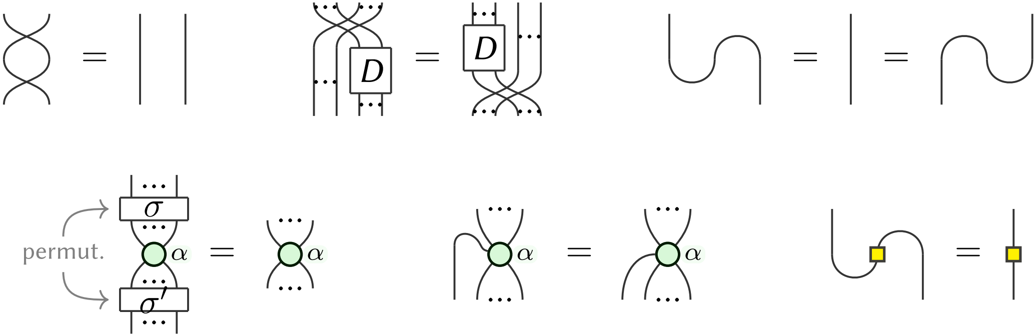
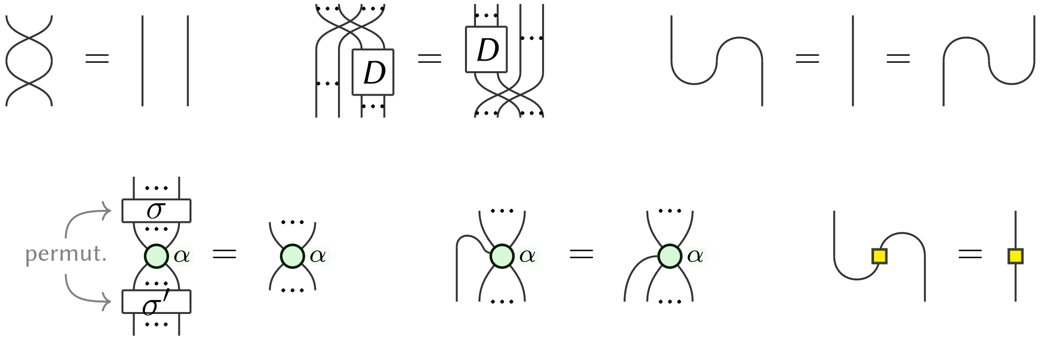
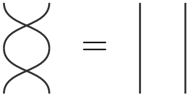
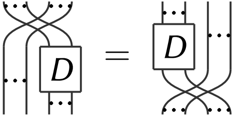


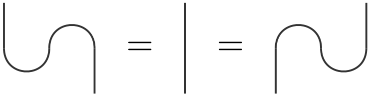


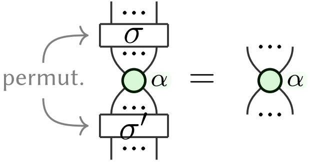
Only Connectivity Matters

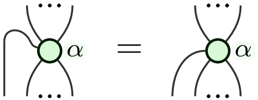
ZX-diagrams can be seen as open graphs. Any graph isomorphism is a valid derivation in the equational theories.

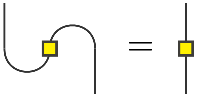












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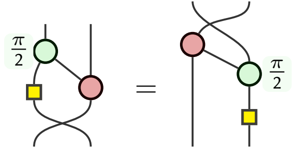
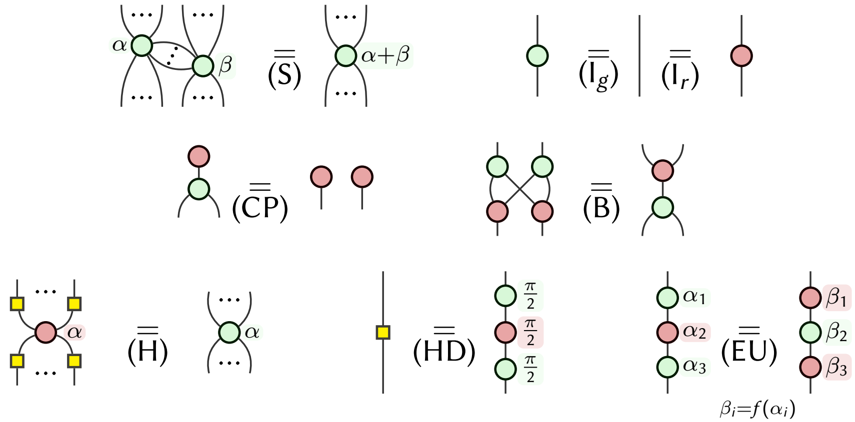
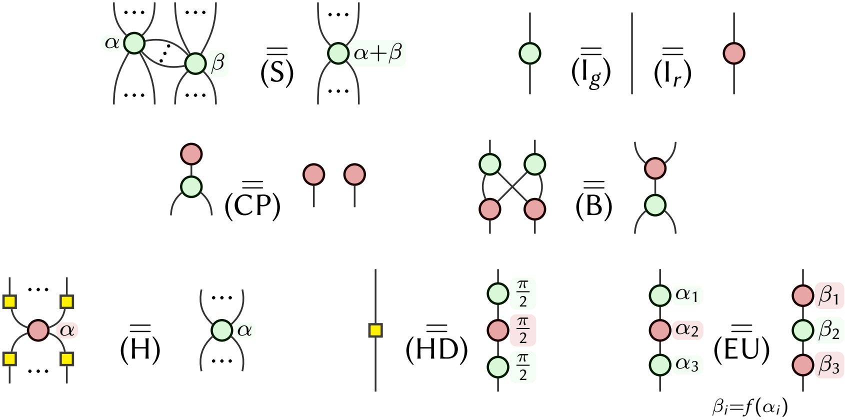
E.g.





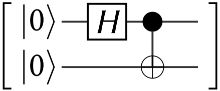
α
β
(S)
α+β
(I_g)
(I_r)
(CP)
(B)
α
(H)
α
(HD)
π/2
π/2
π/2
α1
α2
α3
(EU)
β1
β2
β3
βi=f(αi)



We write $\mathrm{ZX} \vdash D_1 = D_2$. Every colour-swapped rule holds.

The EPR state: $\frac{|00\rangle + |11\rangle}{\sqrt{2}}$

ZX $\vdash$



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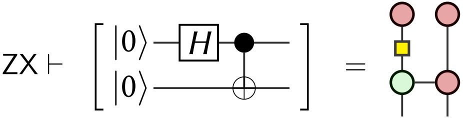

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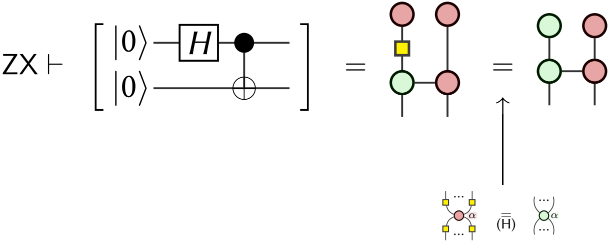

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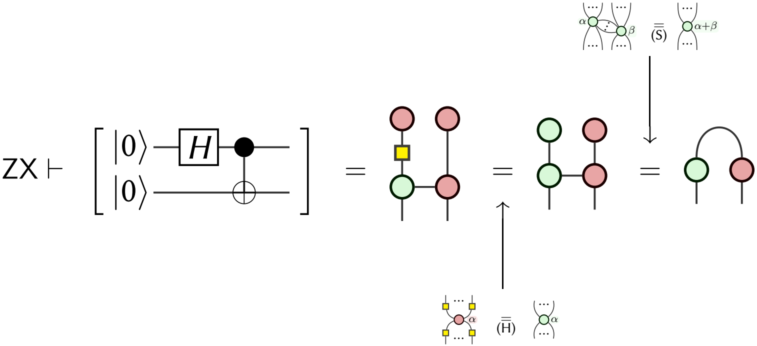

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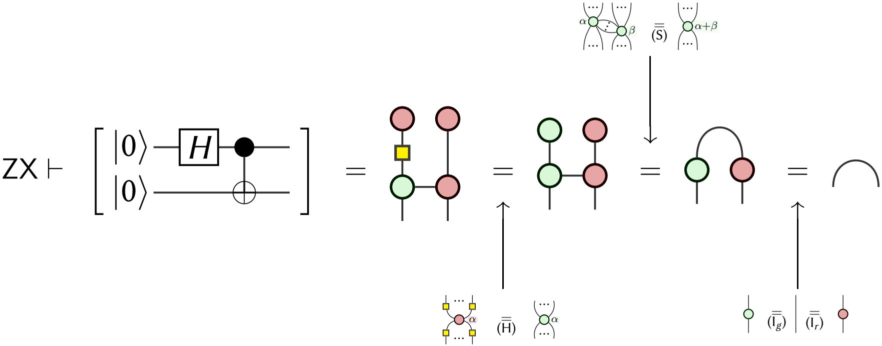

Theorem [V.'19]

The language is *complete*:

$$\forall D_1, D_2 \in \textbf{ZX},\ [\![D_1]\!] = [\![D_2]\!] \iff \text{ZX} \vdash D_1 = D_2$$

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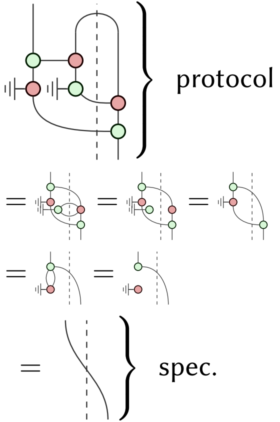
Previous/other completeness results:

- $\frac{\pi}{2}$-fragment [Backens'14]
- π-fragment [Duncan,Perdrix'14]
- 1-qubit $\frac{\pi}{4}$-fragment [Backens'14]
- $\frac{\pi}{4}$-fragment [Jeandel,Perdrix,V.'18]
- full ZX (modified) [Hadzihasanovic,Ng,Wang'18]
- full ZX [Jeandel,Perdrix,V.'18]

Verification



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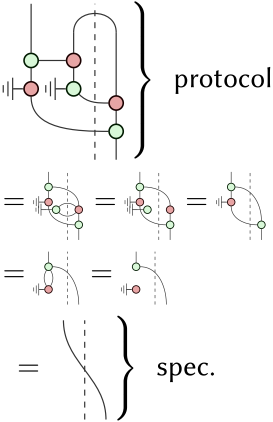
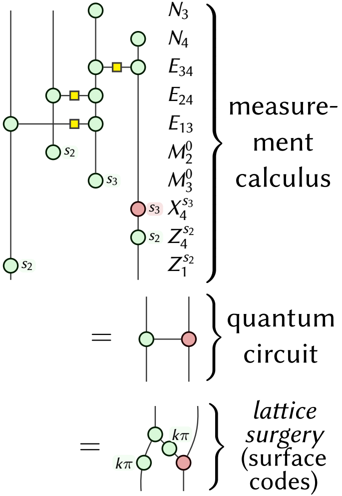
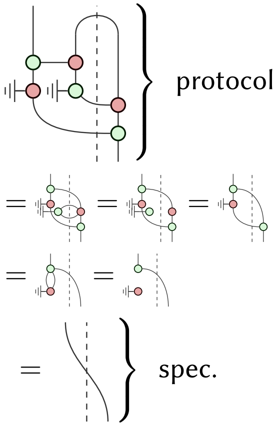


Unification



Verification



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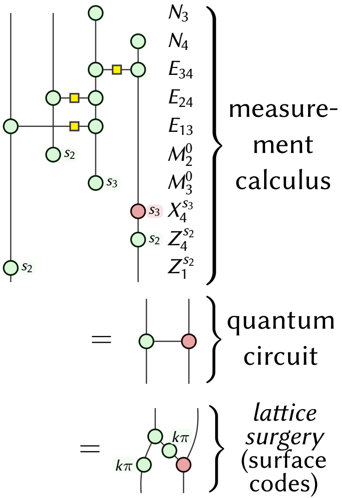
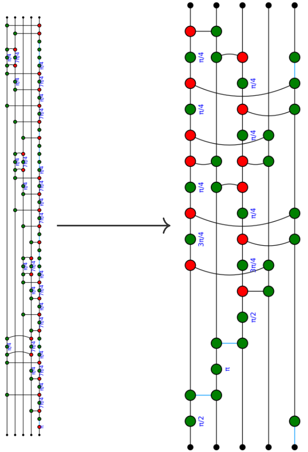


Optimisation



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  - reduces Clifford diagrams to a (pseudo) normal form efficiently
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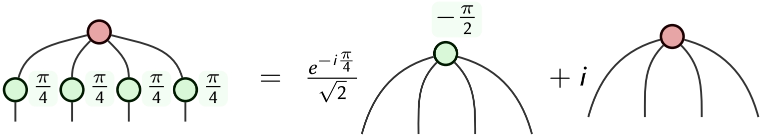
- 

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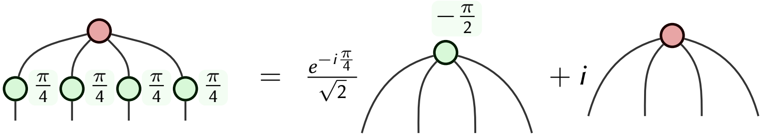
- 

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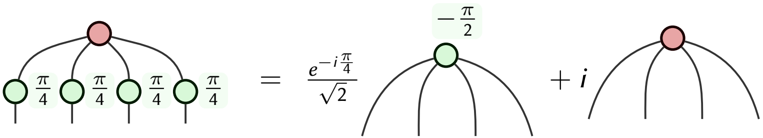
- 

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Simulation [Kissinger,Wetering,V.'22]

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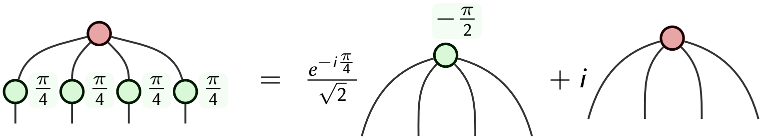
- 

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- Simulation of non-trivial medium-scale circuits

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Conclusion

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- *Formal methods for quantum algorithms: A Survey* [Chareton,Bardin,Lee,Valiron,V.,Xu'21]