First-Order Theorem Proving and Vampire

Laura Kovács

Outline

Automated Reasoning by First-Order Theorem Proving

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2. Automatically checking this statement's consistency or truth

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There are lots of places where we can apply automated reasoning. For example,

- Proving software correctness (partial/total correctness)
- Generating loop invariants
- Program synthesis
- Model checking
- Your idea?

Kinds of Automated Reasoning

Given a statement S we can establish different conclusions about it

- Consistency - there is a way of making S true
- Inconsistency - there is no way of making S true
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We can look at these three notions from two different views.

	Semantic view	Syntactic view
S is consistent	Has a model	No proof of $\perp$ from S
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Notes

1. Here we have focussed only on proofs of inconsistency.
2. Consistency is commonly referred to as satisfiability

Kinds of Automated Reasoners

	Input	Example(s)
SAT Solvers	Propositional formulae	MiniSat
SMT Solvers	(First-order) formulae + theories	Z3,CVC4
Theorem Provers	First-order formulae (+ theories)	Vampire,E
Proof Assistants (interactive)	High-order formulae	Isabelle,Coq

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Above the line focus on models and might be decidable. Below the line focus on proofs and are rarely decidable.

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Getting Started

VAMPIRE: an automated first-order theorem prover

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Go to

`https://vprover.github.io/download.html`

and pick the route most suitable to you.

Notes:

- For Linux users, a binary is probably the easiest route
- For Mac users, you need to build from source
  - run `make vampire_rel`
- For Windows users, the easiest route for this tutorial is a virtual machine and then use Linux

First-Order Theorem Proving. Example

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More formally: in a group "assuming that $x^2 = 1$ for all x prove that $x \cdot y = y \cdot x$ holds for all x, y."
What is implicit: axioms of the group theory.

$$
\begin{array}{l}
\forall x(1 \cdot x = x) \\
\forall x(x^{-1} \cdot x = 1) \\
\forall x \forall y \forall z((x \cdot y) \cdot z = x \cdot (y \cdot z))
\end{array}
$$

Formulation in First-Order Logic

Axioms (of group theory):	$\forall x(1 \cdot x = x)$ $\forall x(x^{-1} \cdot x = 1)$ $\forall x \forall y \forall z((x \cdot y) \cdot z = x \cdot (y \cdot z))$
Assumptions:	$\forall x(x \cdot x = 1)$
Conjecture:	$\forall x \forall y(x \cdot y = y \cdot x)$

In the TPTP Syntax

The TPTP library (Thousands of Problems for Theorem Provers), `http://www.tptp.org` contains a large collection of first-order problems.

For representing these problems it uses the TPTP syntax, which is understood by all modern theorem provers, including Vampire.

In the TPTP Syntax

In the TPTP syntax this group theory problem can be written down as follows:

```
%---- 1 * x = x
fof(left_identity,axiom,
    ! [X] : mult(e,X) = X).
%---- i(x) * x = 1
fof(left_inverse,axiom,
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%---- (x * y) * z = x * (y * z)
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Running Vampire of a TPTP file

is easy: simply use

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One can also run Vampire with various options, some of them will be explained later. For example, save the group theory problem in a file `group.tptp` and try

```
vampire --thanks <your name> group.tptp
```

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- Language: variables, function and predicate (relation) symbols. A constant symbol is a special case of a function symbol.

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- Terms: variables, constants, and expressions $f(t_1, \ldots, t_n)$, where f is a function symbol of arity n and $t_1, \ldots, t_n$ are terms.

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- Atomic formula: expression $p(t_1, \ldots, t_n)$, where p is a predicate symbol of arity n and $t_1, \ldots, t_n$ are terms. Formulas denote properties of domain elements.
- All symbols are uninterpreted, apart from equality $=$.

First-Order Logic and TPTP

FOL	TPTP
$\bot, \top$	`$false`, `$true`
$\neg a$	`~a`
$a_1 \wedge \ldots \wedge\ a_n$	`a1 & ... & an`
$a_1 \vee \ldots \vee\ a_n$	`a1 | ... | an`
$a_1 \rightarrow a_2$	`a1 => a2`
$(\forall x_1)\ldots(\forall x_n)a$	`! [X1,...,Xn] : a`
$(\exists x_1)\ldots(\exists x_n)a$	`? [X1,...,Xn] : a`

More on the TPTP Syntax

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- Comments
- Input formula names

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More on the TPTP Syntax

- Comments
- Input formula names and roles

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More on the TPTP Syntax

- Comments
- Input formula names and roles
- Equality

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Proof by Vampire (Slightly Modified)

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Refutation found.
270. $false [trivial inequality removal 269]
269. mult(sk0,sk1) != mult (sk0,sk1) [superposition 14,125]
125. mult(X2,X3) = mult(X3,X2) [superposition 21,90]
90. mult(X4,mult(X3,X4)) = X3 [forward demodulation 75,27]
75. mult(inverse(X3),e) = mult(X4,mult(X3,X4)) [superposition 22,19]
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22. mult(inverse(X4),mult(X4,X5)) = X5 [forward demodulation 17,10]
21. mult(X0,mult(X0,X1)) = X1 [forward demodulation 15,10]
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21. mult(X0,mult(X0,X1)) = X1 [forward demodulation 15,10]
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Proof by Vampire (Slightly Modified)

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270. $false [trivial inequality removal 269]
269. mult(sk0,sk1) != mult (sk0,sk1) [superposition 14,125]
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Vampire

- Completely automatic: once you started a proof attempt, it can only be interrupted by terminating the process.

Vampire

- Completely automatic: once you started a proof attempt, it can only be interrupted by terminating the process.
- Champion of the CASC world-cup in first-order theorem proving: won CASC $>$ 50 times.

Recap – What an Automatic Theorem Prover is Expected to Do

Input:

- a set of axioms (first order formulas) or clauses;
- a conjecture (first-order formula or set of clauses).

Output:

- proof (hopefully).

Proof by Refutation

Given a problem with axioms and assumptions $F_1, \dots, F_n$ and conjecture G,

1. negate the conjecture;
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In this formulation the negation of the conjecture $\neg G$ is treated like any other formula.

In fact, Vampire (and other provers) internally treat conjectures differently, to make proof search more goal-oriented.

General Scheme (simplified)

- Read a problem;
- Determine proof-search options to be used for this problem;
- Preprocess the problem;
- Convert it into CNF;
- Run a saturation algorithm on it, try to derive *false*.
- If *false* is derived, report the result, maybe including a refutation.

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Trying to derive *false* using a saturation algorithm is the hardest part, which in practice may not terminate or run out of memory.