From Bisimulations to Metrics via Couplings

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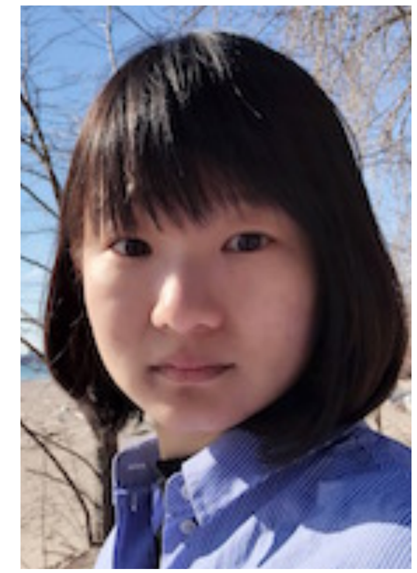
MOVEP 2022 - Aalborg

A big thank to


Giorgio Bacci


Kim G. Larsen

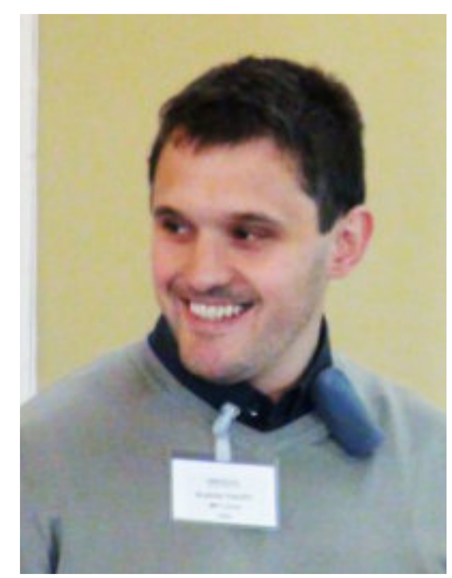

Radu Mardare


Qiyi Tang


Max Tschaikowski


Mirco Tribastone


Franck van Breugel


Andrea Vandin

The Coupling Method

- It's a fundamental proof technique in probability theory
- Used to compare distributions $\mu, \nu \in D(X)$

Main Idea: construct a joint probability $\gamma \in D(X \times X)$ with marginals μ and ν where it's easier to prove the relation

Stochastic Domination & Couplings:

- Assume the set X has an order $\sqsubseteq$
- We write that $\mu \sqsupseteq_{sd} \nu$ iff $\forall a \in X \,.\, \mu[x \sqsupseteq a] \geq \nu[x' \sqsupseteq a]$

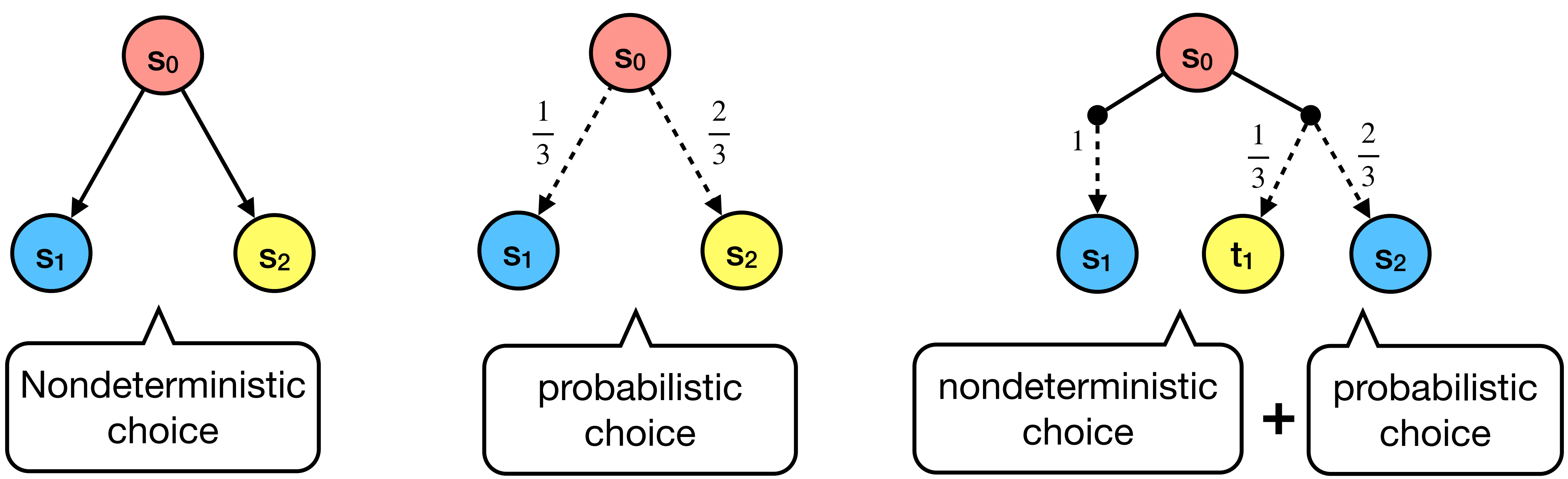
Strassen's Theorem:

$\mu \sqsubseteq_{sd} \nu$ iff $\exists \gamma \in \Gamma_D(\mu, \nu)$ such that $\gamma(x, x') > 0 \implies x \sqsubseteq x'$

Systems' Behaviour & Couplings

- We want to **reason about behaviours** of systems with
  - Nondeterministic choice (e.g. transition systems)
  - Probabilistic choice (e.g., Markov chains)
  - Probabilistic + Nondeterministic choice (e.g., probabilistic automata, Markov decision processes)

  - …and more (spoiler: polynomial ODE)



Equivalences vs. Pseudometrics

Specification $\sim$ Implementation

Specification $\overset{\varepsilon}{\longmapsto}$ Implementation

EQUIVALENCE RELATION:

Reflexive: $s \sim s$

Symmetric: $s \sim t \implies t \sim s$

Transitive: $s \sim u$ and $u \sim t \implies s \sim t$

- Reason about observational equivalence
- Often used to minimise the set of states of the system
- Not informative when the equivalence is not found

PSEUDOMETRIC:

Reflexive: $d(s,s) = 0$

Symmetric: $d(s,t) = d(t,s)$

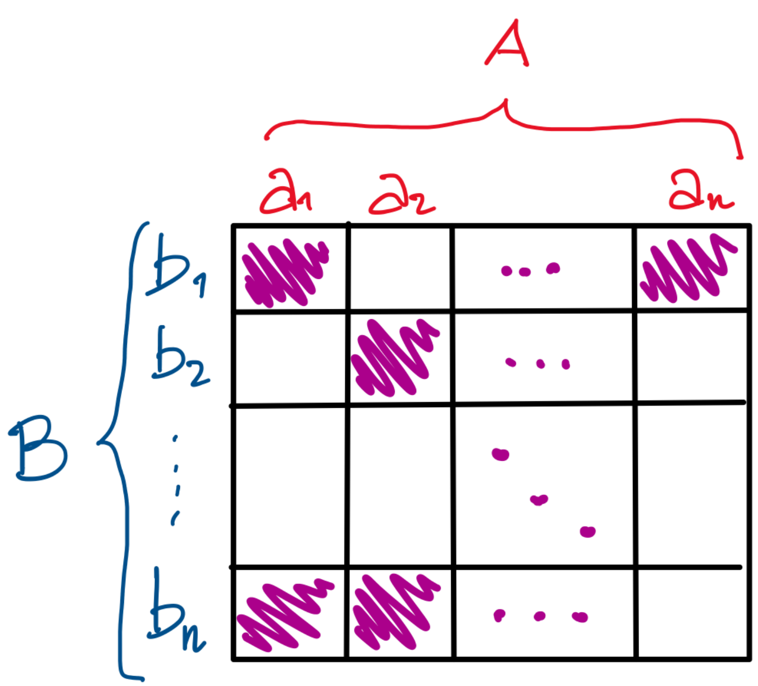
Triangular inequality: $d(s,u) + d(u,t) \leq d(s,t)$

- Measure observational dissimilarities
- May be used to minimise the set of states beyond equivalence
- Provide information about the magnitude of dissimilarity

Two type of Couplings

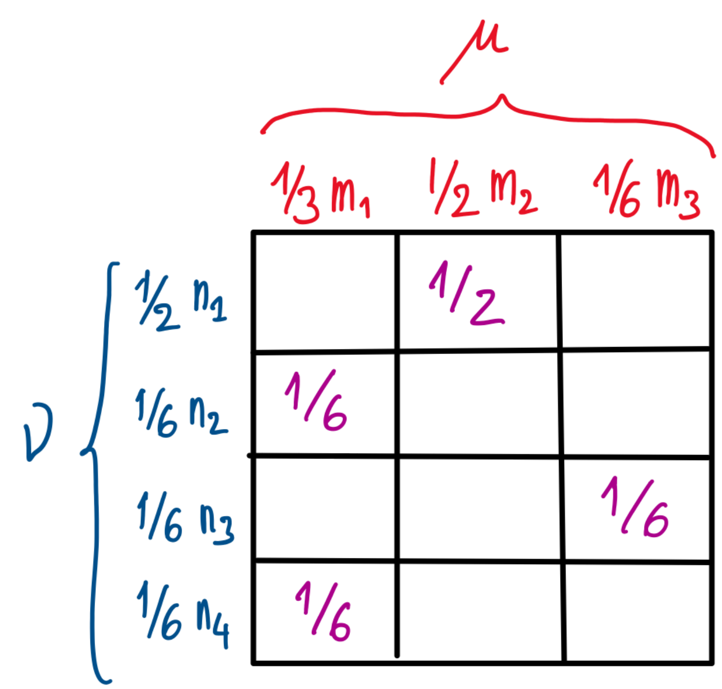
Nondeterministic Coupling

- To relate sets $A, B \subseteq S$
- Here a coupling is a **relation** $\Phi \subseteq A \times B$ such that
  - (i) $A = \{a \in S \colon (a, b) \in \Phi\}$
  - (ii) $B = \{b \in S \colon (a, b) \in \Phi\}$
- We denote $\Gamma_S(A, B)$ the set of nondeterministic couplings for (A,B)



Probabilistic Coupling

- To relate prob. distrib. $\mu, \nu \in D(S)$,
- Here a **coupling** is a probability distribution $\gamma \subseteq D(S \times S)$ such that
  - (i) $\forall s \in S \,.\, \mu(s) = \sum_{t \in S} \gamma(s, t)$
  - (ii) $\forall t \in S \,.\, \nu(t) = \sum_{s \in S} \gamma(s, t)$
- We denote $\Gamma_D(\mu, \nu)$ the set of probabilistic couplings for (μ, ν)



Couplings & Liftings

Nondeterministic Coupling | **Probabilistic Coupling**

Lifting Relations

Lifting Metrics

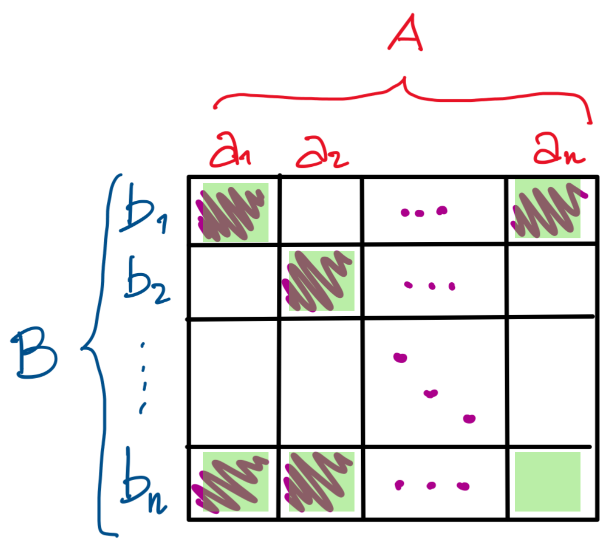
Couplings & Liftings

Nondeterministic Coupling

Probabilistic Coupling

Lifting Relations

$$\mathbf{S}[R] = \{(A, B) \colon \Phi \in \Gamma_{\mathbf{S}}(A, B), \Phi \subseteq R\}$$



Lifting Metrics

Couplings & Liftings

Nondeterministic Coupling

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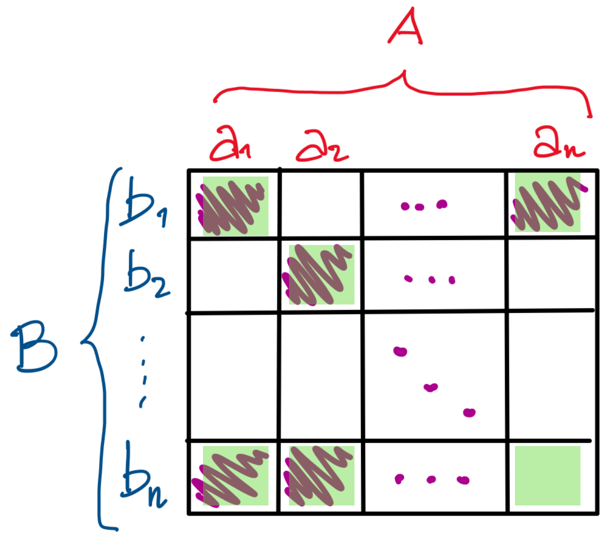
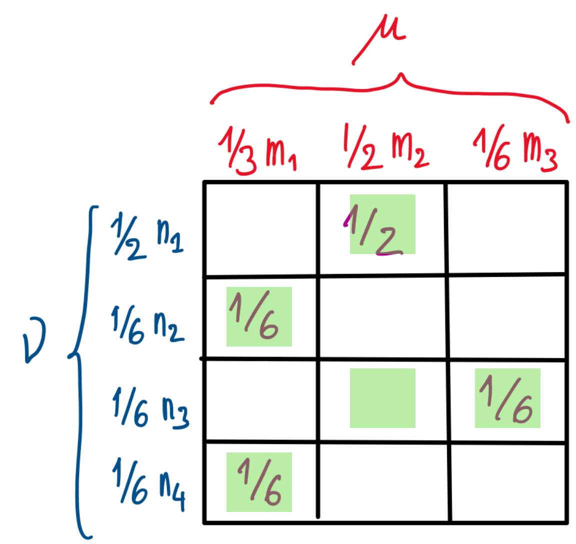


$$\mathbf{D}[R] = \{(\mu, \nu) \colon \gamma \in \Gamma_{\mathbf{D}}(\mu, \nu), supp(\gamma) \subseteq R\}$$



Lifting Metrics

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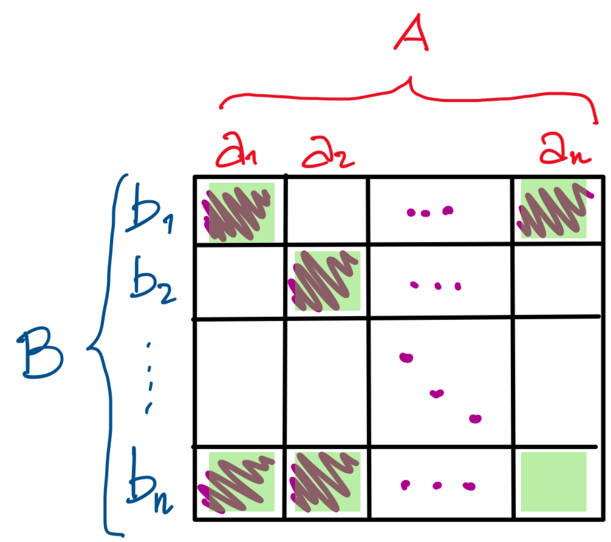

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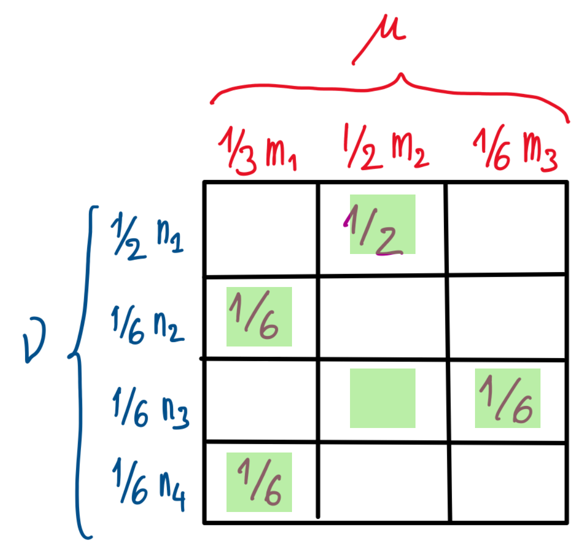

Lifting Metrics

$$\mathbf{H}(d)(A, B) = \min_{\Phi \in \Gamma_S(A,B)} \max_{(a,b)\in\Phi} d(a, b)$$

Hausdorff distance!
(see Mémoli'11)

Couplings & Liftings

Nondeterministic Coupling

Probabilistic Coupling

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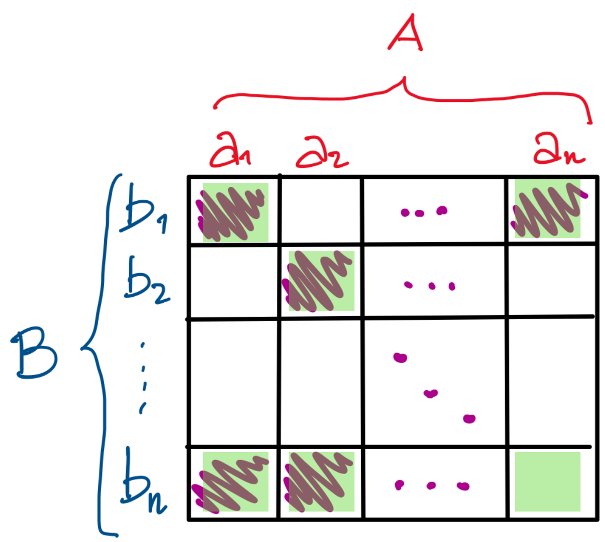

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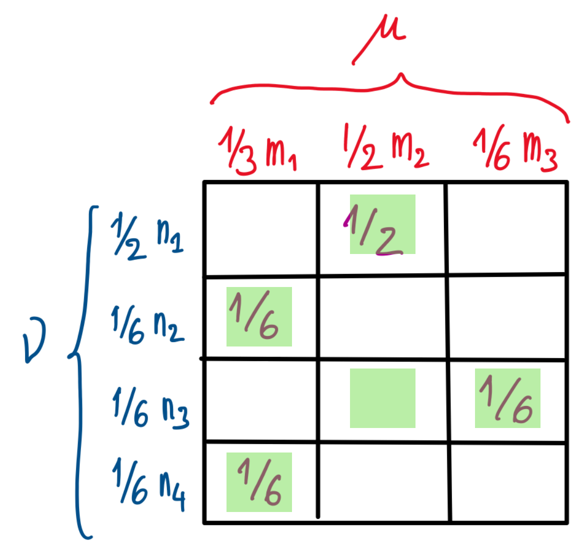


Lifting Metrics

$$\mathbf{H}(d)(A,B) = \min_{\Phi \in \Gamma_S(A,B)} \max_{(a,b)\in\Phi} d(a,b)$$

Hausdorff distance!
(see Mémoli'11)

$$\mathbf{K}(d)(\mu,\nu) = \min_{\gamma \in \Gamma_D(\mu,\nu)} \sum_{s,t\in S} d(s,t)\cdot\gamma(s,t)$$

Kantorovich distance!

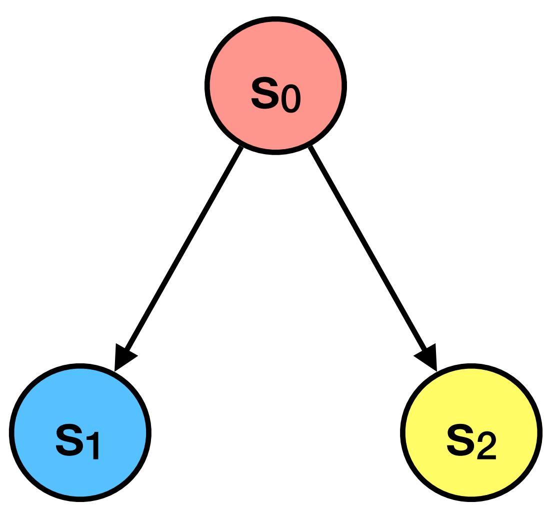
Transition Systems

$$\mathcal{T} = (S, \delta, \ell)$$

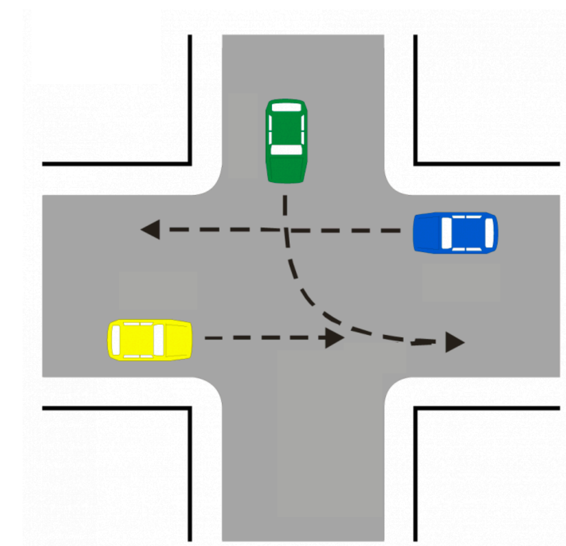
States

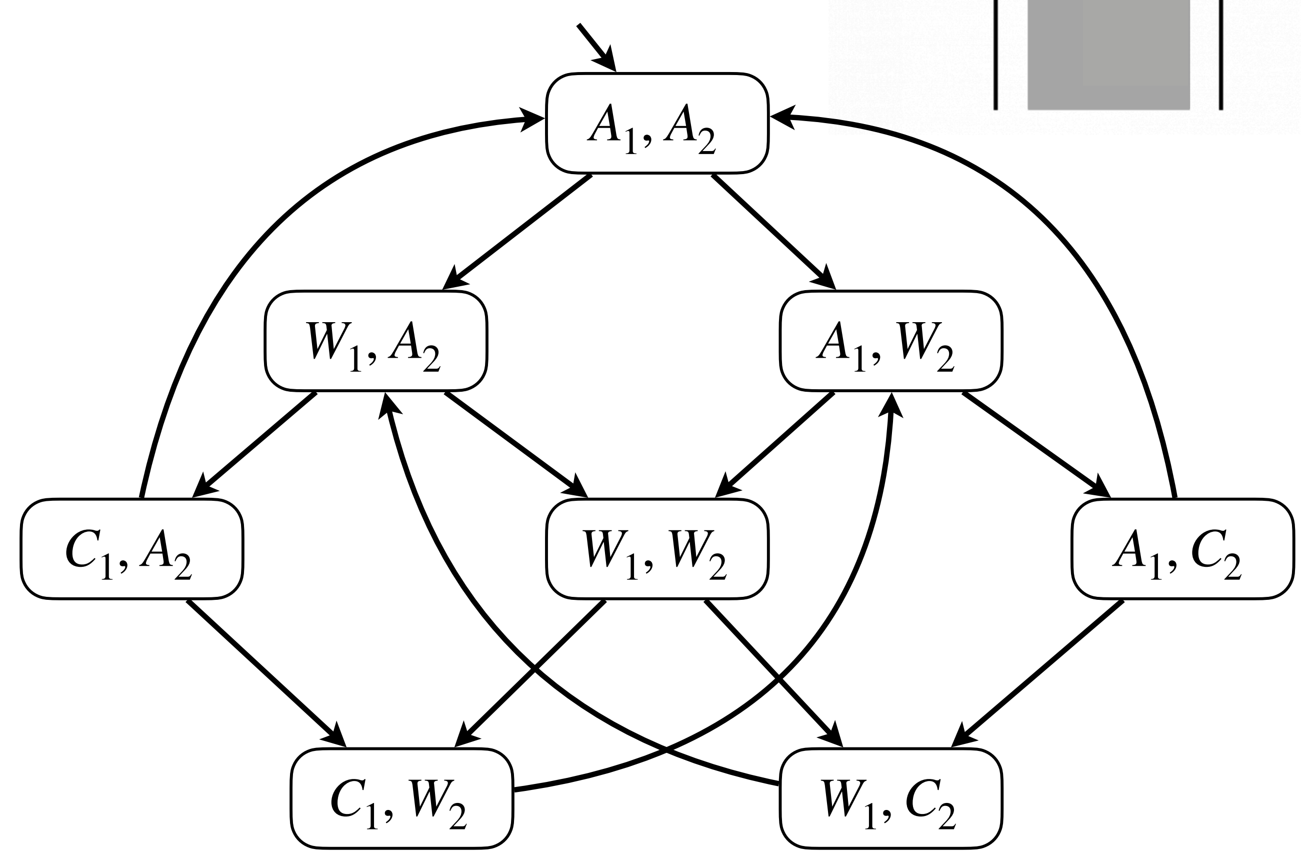
Successor function $\delta\colon S \to 2^S$

Labelling function $\delta\colon S \to L$



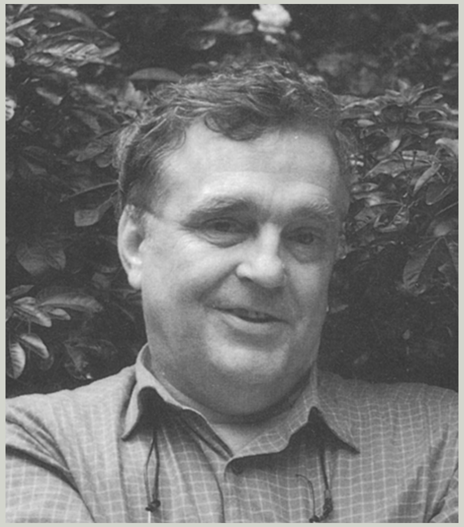
Nondeterministic choices





Bisimulation

- Initially formulated by Robin Milner under the name "observation equivalence" in 1980
- Perfected by David Park with a fixed point characterisation.



Definition:

$R \subseteq S \times S$ is a *bisimulation* if whenever $(s, t) \in R$ then

(i) **What we observe in the two states is the same**, i.e., $\ell(s) = \ell(t)$

(ii) **Each transition of one can be matched by some transition of the other and vice versa, formally**:

- $\forall s' \in \delta(s)\ \exists t' \in \delta(t)$ such that $(s', t') \in R$
- $\forall t' \in \delta(t)\ \exists s' \in \delta(s)$ such that $(s', t') \in R$

Fixed point characterisation

...just rephrasing Park's idea

$$\mathscr{B}(R) = \big\{(s,t) \in S \times S \mid (\delta(s), \delta(t)) \in S[R]\big\}$$

$$\begin{tikzcd}
s \arrow[r, dash, "R"] \arrow[d, mapsto] & t \arrow[d, mapsto] \\
\delta(s) \arrow[r, dash, "{S[R]}"'] & \delta(t)
\end{tikzcd}$$

Theorem (Fixed point):

For any $R \subseteq S \times S$

(1) $R \subseteq \mathscr{B}(R)$ iff R is a bisimulation

(2) gfp$(\mathscr{B})$ coincides with bisimilarity

$\mathbf{S}[R] = \{(A,B) \colon \Phi \in \Gamma_{\mathbf{S}}(A,B)$ and $\Phi \subseteq R\}$

Coupled system

Given the TS $\mathcal{T}$, a $\mathcal{T}^2 = (S^2, \delta^2, \{eq, neq\}, \ell^2)$ is a "coupled" system for $\mathcal{T}$ if

(i) $\delta^2(s,t) \in \Gamma_S(\delta(s), \delta(t))$ for all $s, t \in S$

(ii) $\ell^2(s,t) = eq$ if $\ell(s) = \ell(t)$; $\ell(s,t) = neq$ otherwise

$$\begin{tikzcd}
s \arrow[d, mapsto] & (s,t) \arrow[l, mapsto, "\pi_1"'] \arrow[r, mapsto, "\pi_2"] \arrow[d, mapsto] & t \arrow[d, mapsto] \\
\delta(s) & \delta^2(s,t) \arrow[l, mapsto, "\pi_1"'] \arrow[r, mapsto, "\pi_2"] & \delta(t)
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Theorem:

$s \sim t$ iff $\mathcal{T}^2, (s,t) \models \Box\, eq$ for some coupled system $\mathcal{T}^2$

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(ii) $\ell^2(s,t) = eq$ if $\ell(s) = \ell(t)$; $\ell(s,t) = neq$ otherwise

$$(s,t) \longrightarrow (s_1, t_1) \longrightarrow (s_2, t_2) \longrightarrow (s_3, t_3) \longrightarrow \cdots$$

Theorem:

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Coupled system

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(ii) $\ell^2(s,t) = eq$ if $\ell(s) = \ell(t)$; $\ell(s,t) = neq$ otherwise

$$
\begin{tikzcd}
t \arrow[r] & t_1 \arrow[r] & t_2 \arrow[r] & t_3 \arrow[r] & \cdots \\
(s,t) \arrow[r] \arrow[u] \arrow[d] & (s_1,t_1) \arrow[r] \arrow[u] \arrow[d] & (s_2,t_2) \arrow[r] \arrow[u] \arrow[d] & (s_3,t_3) \arrow[r] \arrow[u] \arrow[d] & \cdots \\
s \arrow[r] & s_1 \arrow[r] & s_2 \arrow[r] & s_3 \arrow[r] & \cdots
\end{tikzcd}
$$

Theorem:

$s \sim t$ iff $\mathscr{T}^2, (s,t) \vDash \Box\, eq$ for some coupled system $\mathscr{T}^2$

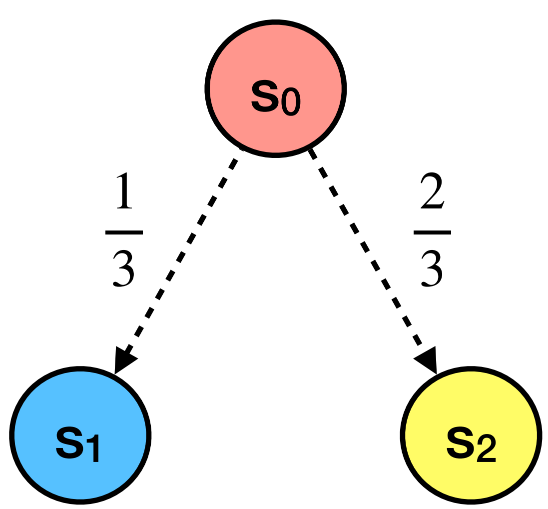
Markov chains

$$\mathscr{M} = (S, \tau, \ell)$$

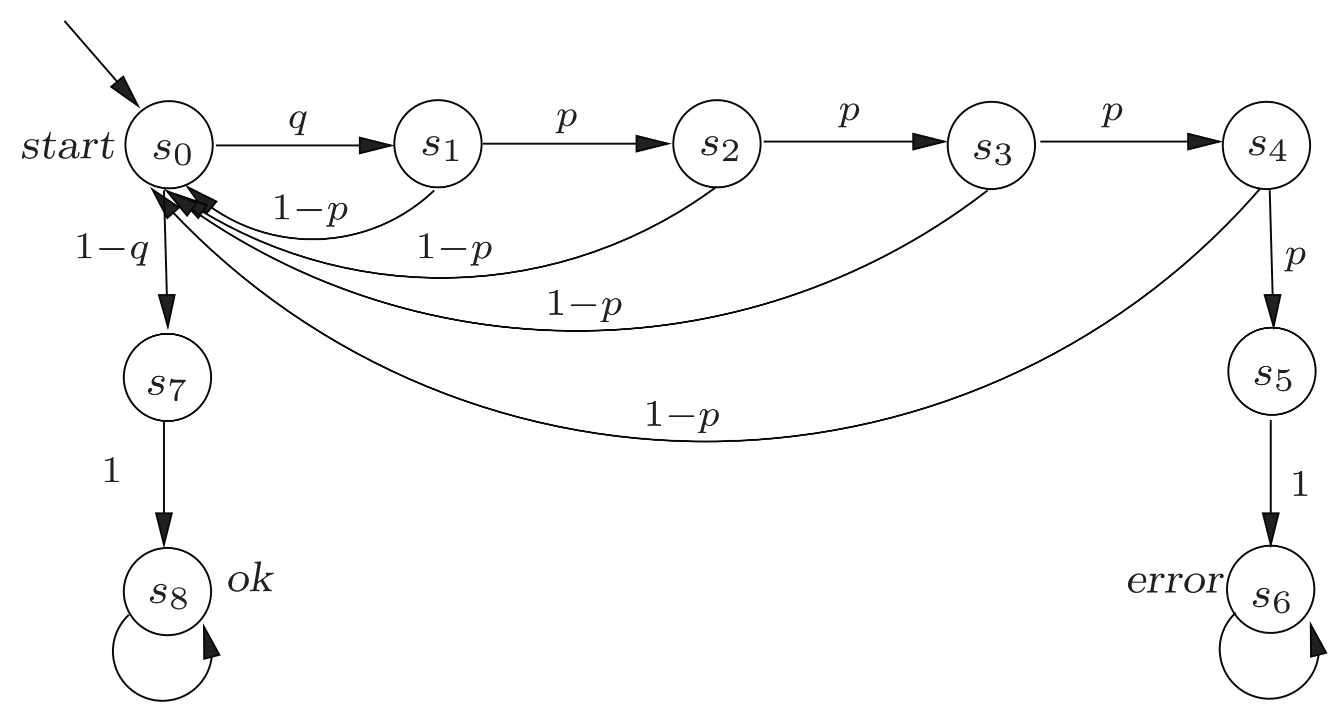
States

Successor function $\tau : S \to D(S)$

Labelling function $\ell : S \to L$



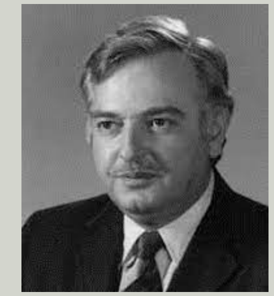
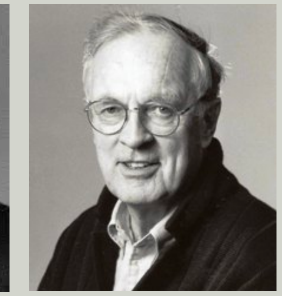
probabilistic choices



Markov chain of the IPv4 zeroconf protocol (for n=4 probes) where $p, q \in (0,1)$. Figure from "Principles of Model Checking" by C. Baier & J-P. Katoen

Probabilistic Bisimulation

- Initially formulated by Kemeny and Snell under the name “**lumpability**”
- Larsen & Skou characterise it via “**probabilistic testability**”

Definition:

Let $R \subseteq S \times S$ be an equivalence relation, then is a *probabilistic bisimulation* if whenever $(s, t) \in R$ then

(i) What we observe in the two states is the same, i.e., $\ell(s) = \ell(t)$

(ii) The probability to move to R-equivalent states is the same, i.e., $\forall C \in S/_R \ \sum_{c \in C} \tau(s)(c) = \sum_{c \in C} \tau(t)(c)$

Fixed point characterisation

…just rephrasing Jonsson & Larsen'91

$$\mathscr{B}(R) = \Big\{ (s,t) \in S \times S \mid (\tau(s), \tau(t)) \in D[R] \Big\}$$

$$\begin{tikzcd} s \arrow[r, dash, "R"] \arrow[d, mapsto] & t \arrow[d, mapsto] \\ \tau(s) \arrow[r, dash, "{D[R]}"'] & \tau(t) \end{tikzcd}$$

$\mathbf{D}[R] = \{(\mu, \nu) \colon \gamma \in \Gamma_{\mathbf{D}}(\mu, \nu)$ and $supp(\gamma) \subseteq R\}$

Theorem (Fixed point):
For any $R \subseteq S \times S$

(1) $R \subseteq \mathscr{B}(R)$ iff R is a bisimulation

(2) gfp($\mathscr{B}$) coincides with bisimilarity

Remark: Baier'96 used the above characterisation to show that probabilistic bisimulation can be computed in polynomial time

Coupled Markov chain

Given the MC $\mathscr{M}$, a $\mathscr{M}^2 = (S^2, \tau^2, \{eq, neq\}, \ell^2)$ is a “coupled” Markov chain for $\mathscr{M}$ if

(i) $\tau^2(s,t) \in \Gamma_D(\tau(s), \tau(t))$ for all $s, t \in S$

(ii) $\ell^2(s,t) = eq$ if $\ell(s) = \ell(t)$; $\ell(s,t) = neq$ otherwise

$$\begin{tikzcd}
s \arrow[d, mapsto] & (s,t) \arrow[l, mapsto, "\pi_1"'] \arrow[r, mapsto, "\pi_2"] \arrow[d, mapsto] & t \arrow[d, mapsto] \\
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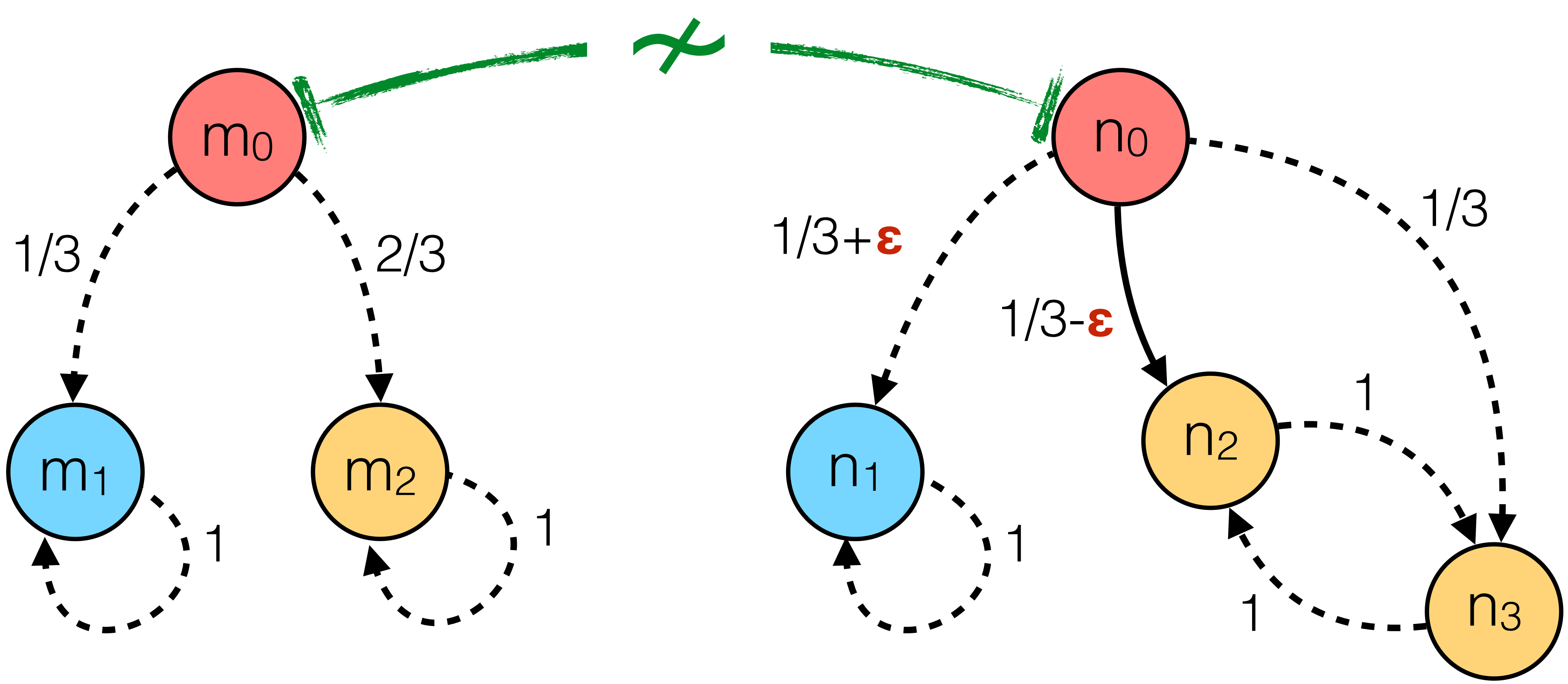
Theorem [Chen, van Breugel, Worrell’12]

$s \sim t$ iff $P[\mathscr{M}^2, (s,t) \vDash \diamond\, neq] = 0$ for some coupled chain $\mathscr{M}^2$

From Bisimulations to Metrics

Jou & Smolka'90 observed that behavioural equivalences are not robust for systems with real-valued data



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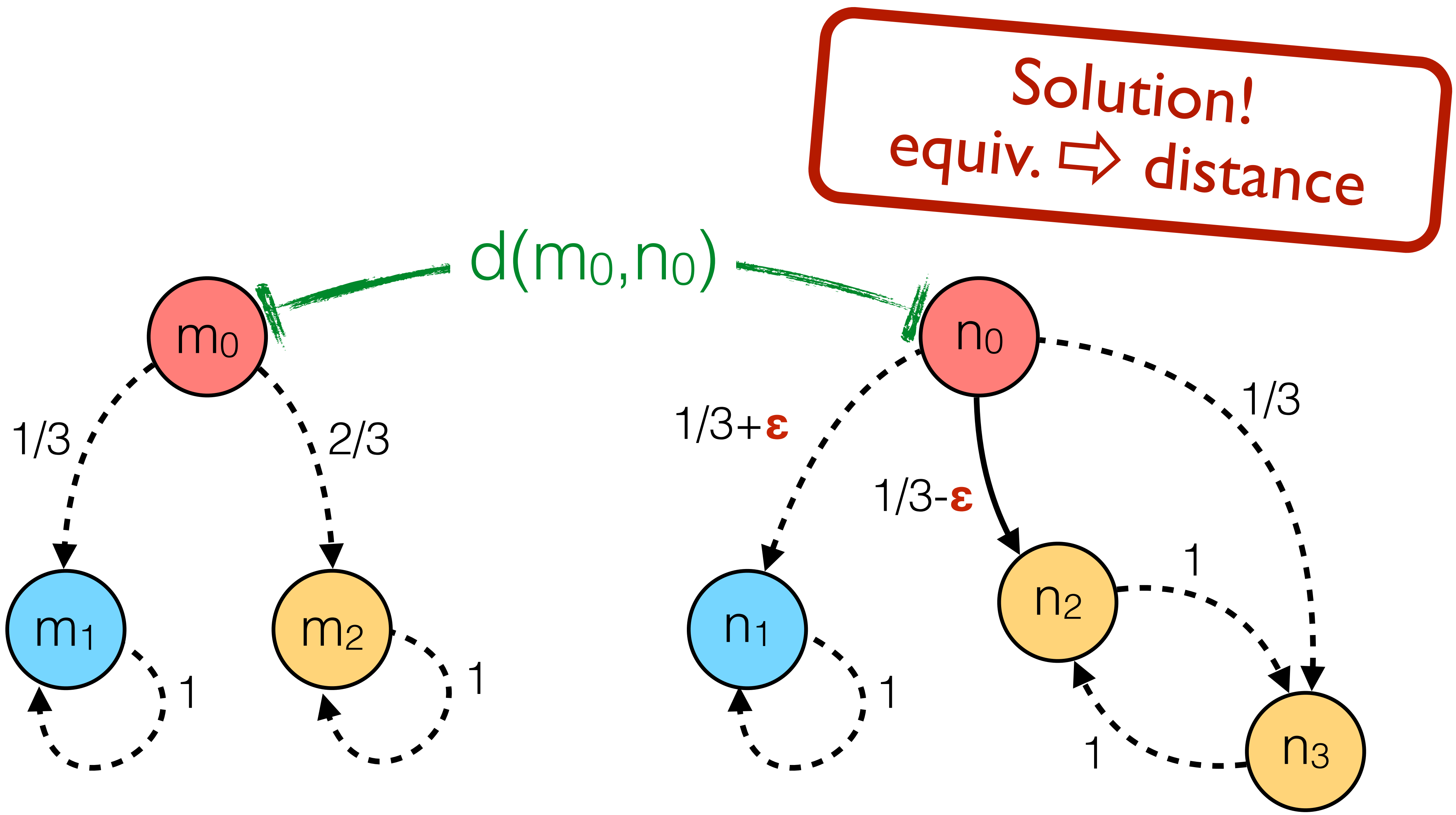

Probabilistic Bisimilarity Distance

- First formulated by Desharnais, Gupta, Jagadeesan, and Panangaden
- Then, van Breugel and Worrell gave a fixed point characterisation






Fixed point characterisation:

The bisimilarity distance $\mathbf{d}_b : S \times S \to [0,1]$ is the least fixed point of the following (monotone) operator

$$\Delta(d)(s,t) = \begin{cases} 1 & \text{if } \ell(s) \neq \ell(t) \\ \mathscr{K}(d)(\tau(s), \tau(t)) & \text{otherwise} \end{cases}$$

Kantorovich distance between transition prob.

Coupled Markov chain (part 2)

$$
\begin{tikzcd}
s \arrow[d, mapsto] & (s, t) \arrow[l, mapsto, "\pi_1"'] \arrow[r, mapsto, "\pi_2"] \arrow[d, mapsto] & t \arrow[d, mapsto] \\
\tau(s) & \tau^2(s, t) \arrow[l, mapsto, "\pi_1"'] \arrow[r, mapsto, "\pi_2"] & \tau(t)
\end{tikzcd}
$$

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\end{tikzcd}
$$

Coupling Theorem [Chen, van Breugel, Worrell'12]

(i) $\mathbf{d}_b(s,t) \leq P[\mathcal{M}^2, (s,t) \models \diamond\, neq]$ for all coupled chain $\mathcal{M}^2$

(ii) $\mathbf{d}_b(s,t) = P[\mathcal{M}^2, (s,t) \models \diamond\, neq]$ for some coupled chain $\mathcal{M}^2$

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(ii) $\mathbf{d}_b(s,t) = P[\mathscr{M}^2, (s,t) \models \diamond\, neq]$ for some coupled chain $\mathscr{M}^2$

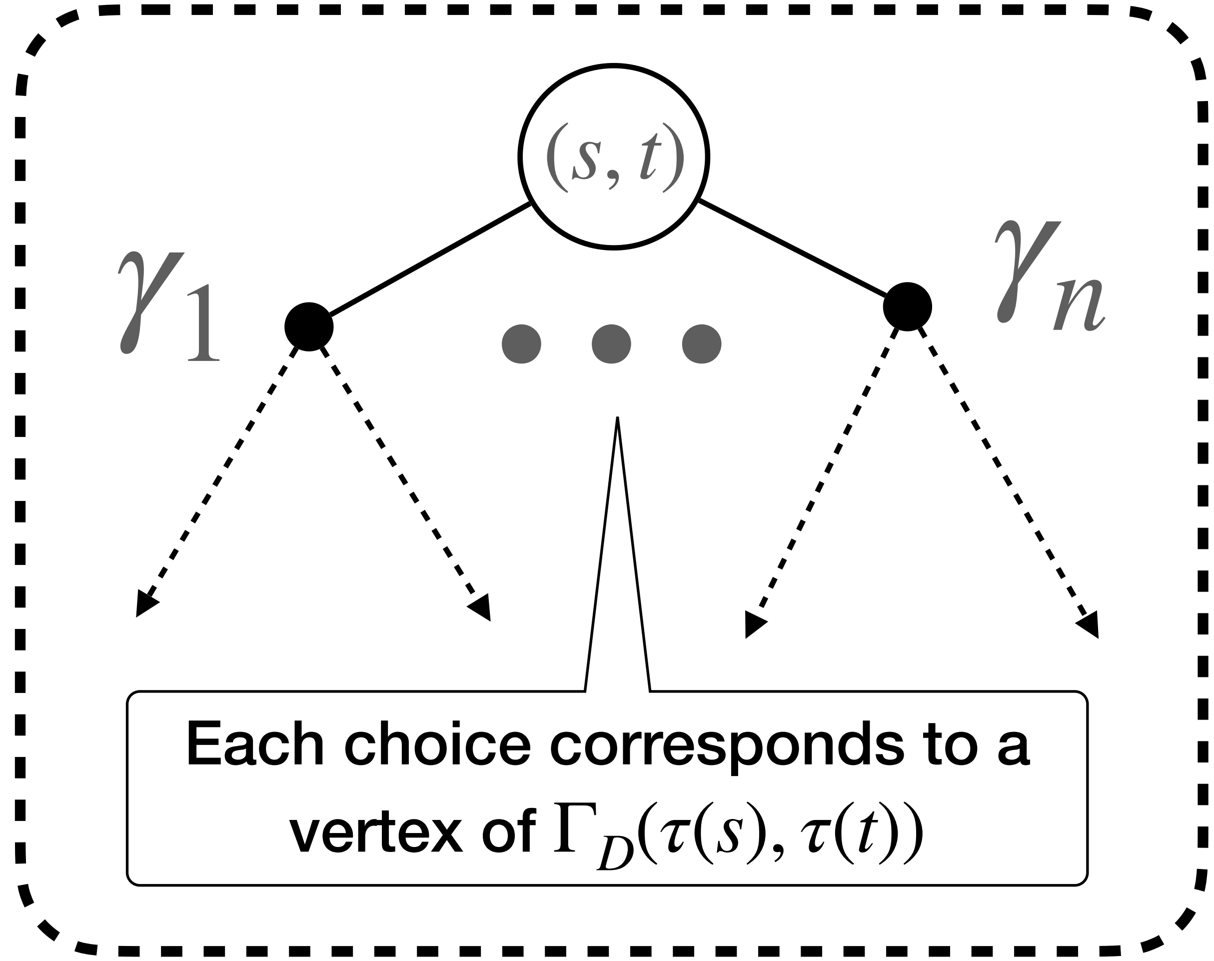
Nice behavioural properties:

(i) $\mathbf{d}_b(s,t) = 0$ iff $s \sim t$

(ii) $sup_{\phi \in LTL} |\, P[\mathscr{M}, s \models \phi] - P[\mathscr{M}, t \models \phi]\,| \leq \mathbf{d}_b(s,t)$

Bisim. Distance & Optimal value

Define a (coupled) Markov decision process $\mathscr{C}$ as follows



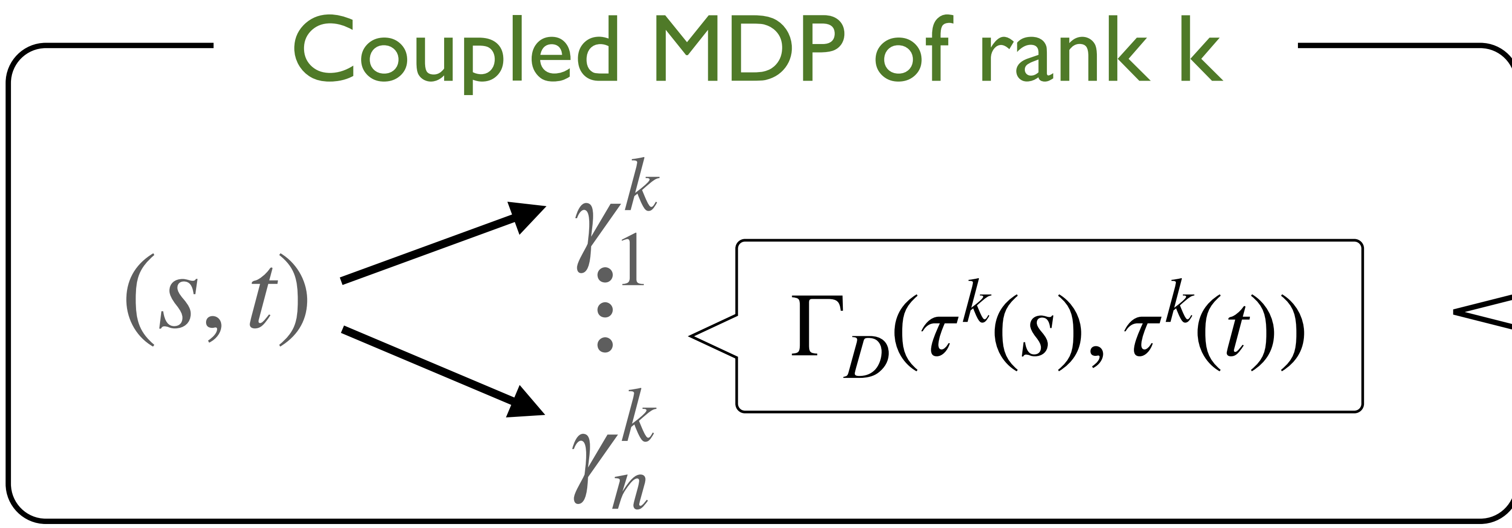
Theorem [Bacci[2], Larsen, Mardare'13]

$$\mathbf{d}_b(s,t) = \inf_{\pi \in \Pi} P^{\pi}[\mathscr{C}, (s,t) \models \diamond\, neq]$$

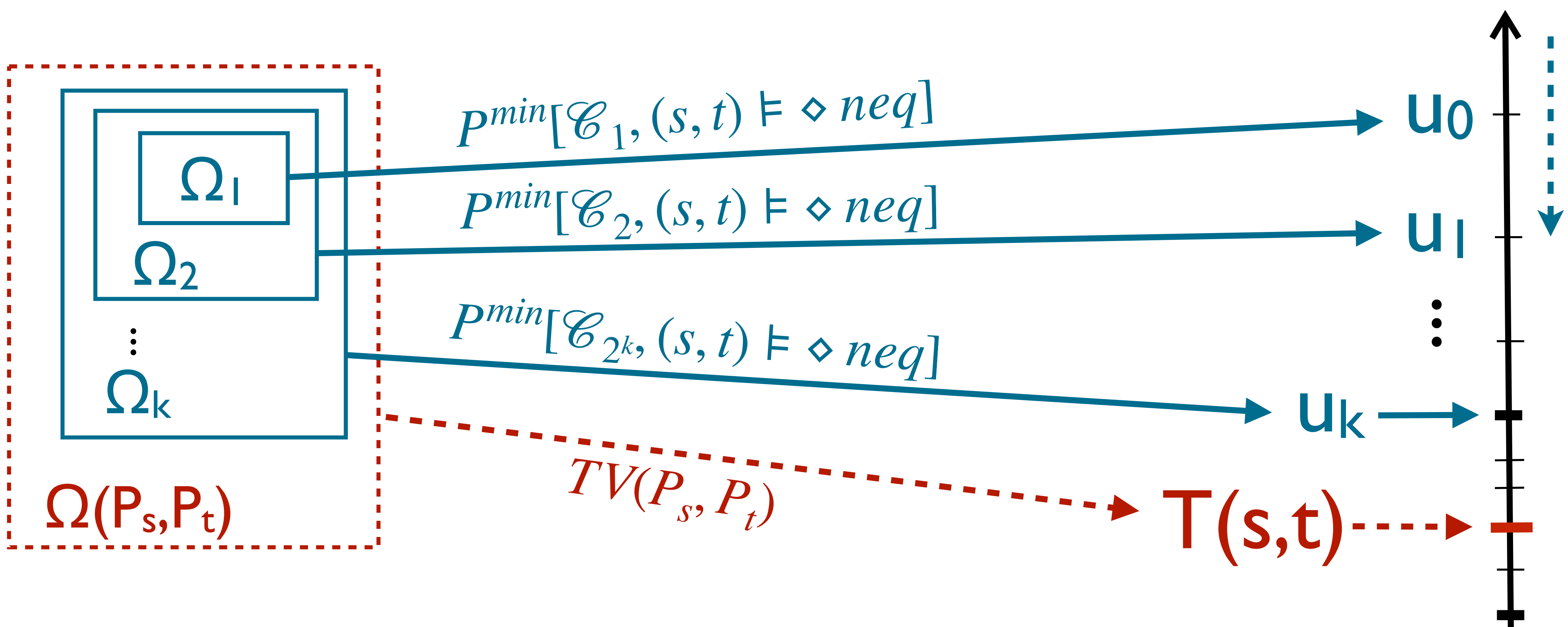
We proposed an <u>on-the-fly</u> policy iteration procedure to compute $\mathbf{d}_b(s,t)$ (see Bacci et al. TACAS'13)

Approximating Total Variation

[Bacci², Larsen, Mardare ICTAC'15]



MDP emitting pairs of **k** **successive steps** at each time interval



$$u_k = P^{min}[\mathscr{C}_{2^k}, (s,t) \models \diamond\, neq]$$

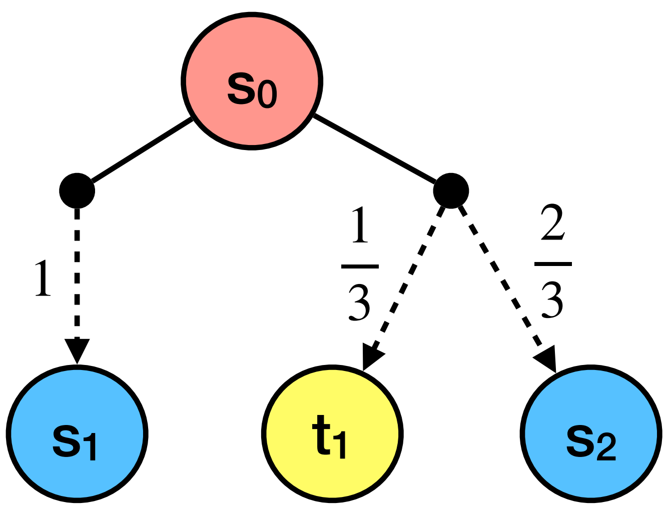
Probabilistic Automata

$$\mathscr{A} = (S, \delta, \ell)$$

States

Successor function $\tau: S \to 2^{D(S)}$

Labelling function $\ell: S \to L$



nondeterministic choice + probabilistic choices

Remark: similar to MDPs but here the nondeterministic choice is taken internally by the system

Probabilistic Bisimilarity Distance

- Generalises bisimilarity distance by Segala and Lynch
- Introduced by Deng, Chothia, Palamidessi, and Pang

Definition:

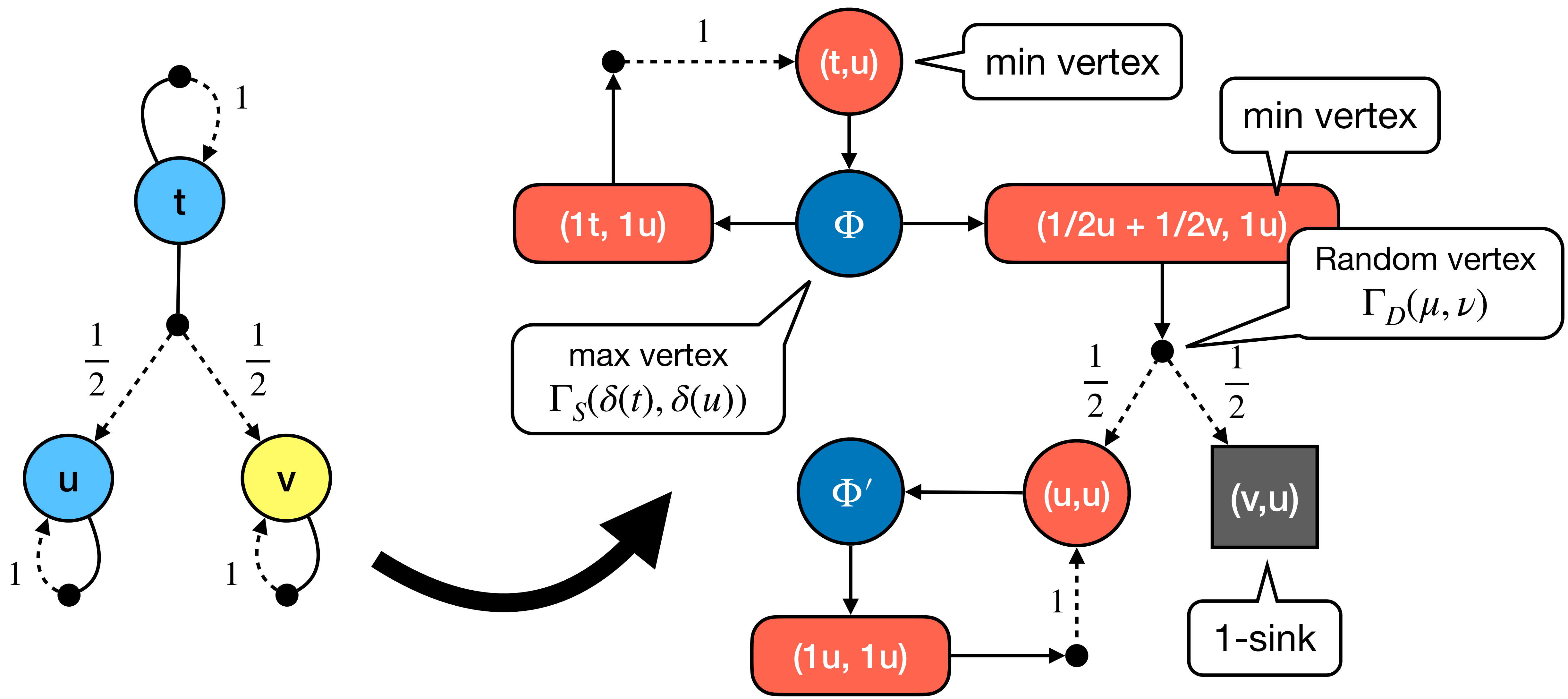
The bisimilarity distance $\mathbf{d}_b : S \times S \to [0,1]$ is the least fixed point of the following (monotone) operator

$$\Delta(d)(s,t) = \begin{cases} 1 & \text{if } \ell(s) \neq \ell(t) \\ \mathscr{H}(\mathscr{K}(d))(\delta(s), \delta(t)) & \text{otherwise} \end{cases}$$

Compose Hausdorff and Kantorovich lifting

Prob. Bisimilarity Game

[Bacci², Larsen, Mardare, Tang, van Breugel'19]



Prob. Bisimilarity Game

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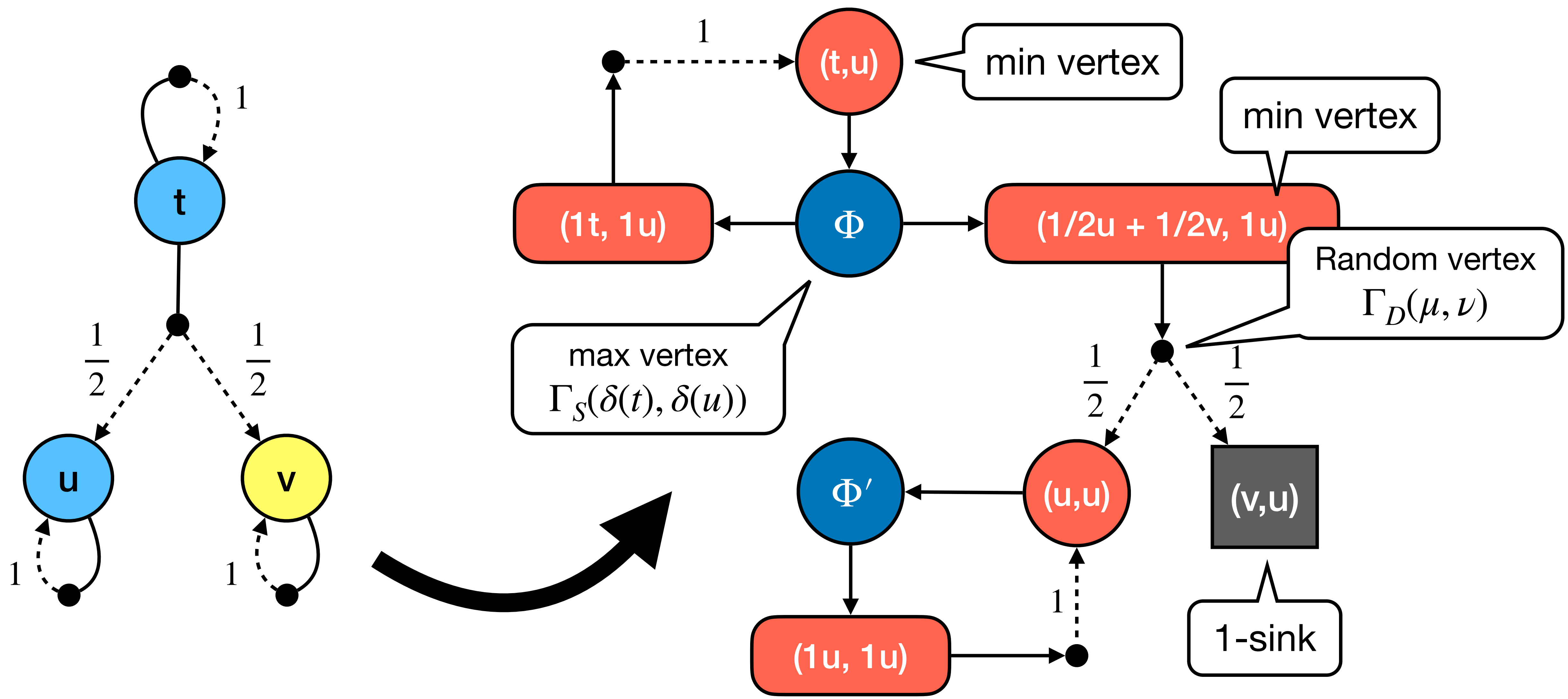


Theorem [Bacci², Larsen, Mardare, Tang, van Breugel'19]

Let G be the SSGs induced by $\mathscr{A}$. Then, the optimal value of the equals $\mathbf{d}_b$

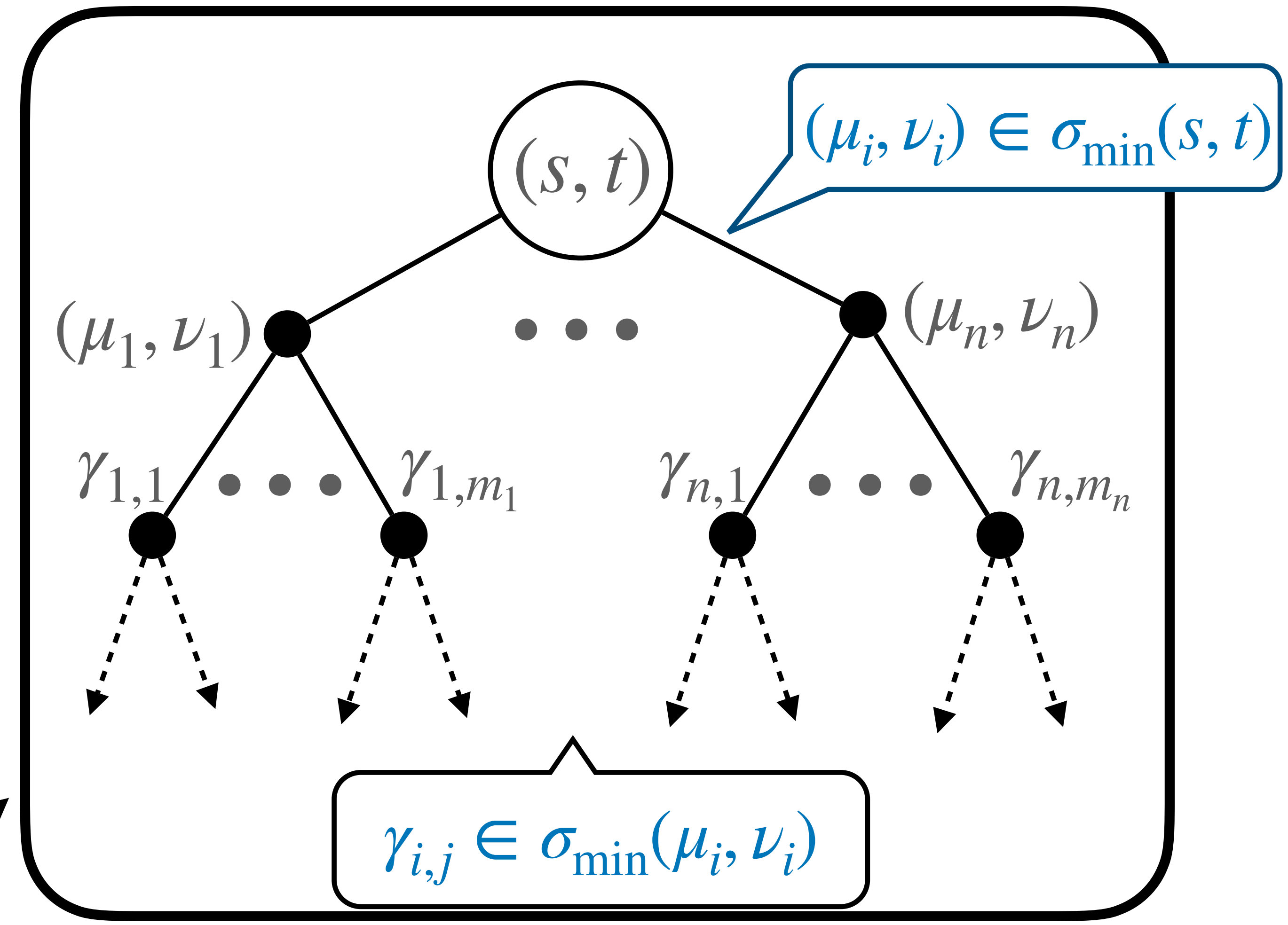
Coupled Probabilistic Automata

[Bacci[2], Larsen, Mardare, Tang, van Breugel'19]

A strategy for the min-player

- $\sigma_{\min}(s, t) \in \Gamma_S(\delta(s), \delta(t))$
- $\sigma_{\min}(\mu, \nu) \in \Gamma_D(\mu, \nu)$

…induces a coupled probabilistic automaton $\mathscr{A}^2$



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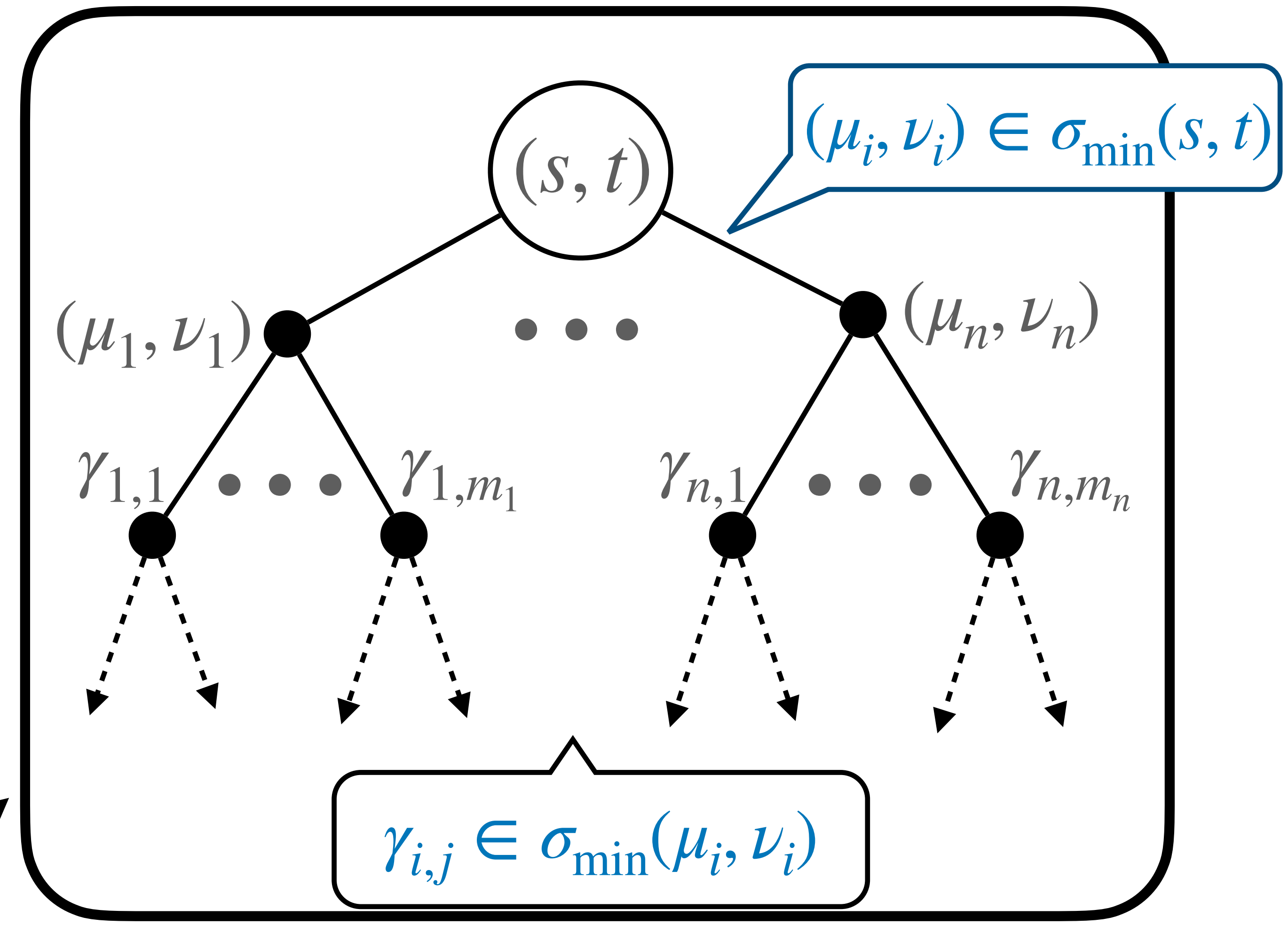

Theorem [Bacci^2, Larsen, Mardare, Tang, van Breugel'21]

(i) $\mathbf{d}_b(s,t) \leq \sup_{\pi\in\Pi} P^\pi[\mathscr{A}^2, (s,t) \vDash \diamond\, neq]$ for all coupled automata $\mathscr{A}^2$

(ii) $\mathbf{d}_b(s,t) = \sup_{\pi\in\Pi} P^\pi[\mathscr{A}^2, (s,t) \vDash \diamond\, neq]$ for some coupled automaton $\mathscr{A}^2$

Relation with Model Checking

[Bacci[2], Larsen, Mardare, Tang, van Breugel'21]

Some useful upper-bounds w.r.t. linear-time model checking

Theorem: For any LTL formula φ,

$$|Max_s(\varphi) - Max_t(\varphi)| \leq \mathbf{d}_b(s,t) \quad \text{and} \quad |Min_s(\varphi) - Min_t(\varphi)| \leq \mathbf{d}_b(s,t)$$

where $Max_s(\varphi) = sup_{\pi \in \Pi} P^{\pi}[\mathscr{A}, s \vDash \varphi]$ and $Min_s(\varphi) = \inf_{\pi \in \Pi} P^{\pi}[\mathscr{A}, s \vDash \varphi]$

Theorem:

$$\mathscr{H}(\mathbb{TV})\Big(\{P^{\pi}_s \mid \pi \in \Pi\}, \{P^{\pi}_t \mid \pi \in \Pi\}\Big) \leq \mathbf{d}_b(s,t)$$

where $\mathbb{TV}(\mu, \nu) = \sup_{\varphi \in LTL} |\mu(\varphi) - \nu(\varphi)|$

$$\forall \pi \in \Pi \,.\, \exists \pi' \in \Pi \,.\, |P^{\pi}[\mathscr{A}, s \vDash \varphi] - P^{\pi'}[\mathscr{A}, t \vDash \varphi]| \leq \mathbf{d}_b(s,t)$$

Bonus Material

…if someone is still awake

Bisimulations for ODEs

[Cardelli, Tribastone, Tschaikowski, Vandin'16]

$$\dot{x}_1 = f_{x_1}(x_1, \ldots, x_n)$$
$$\dot{x}_2 = f_{x_2}(x_1, \ldots, x_n)$$
$$\vdots$$
$$\dot{x}_n = f_{x_n}(x_1, \ldots, x_n)$$

ODE system induced by a polynomial vector field $f\colon \mathbb{R}^X \to \mathbb{R}^X$

Backward Differential Equivalence (BDE)

Definition 1 (Backward differential equivalence). *Let f be a vector field over X. An equivalence relation $R \subseteq X \times X$ is a BDE for f if the implication*

$$\Big(\bigwedge_{(x,y)\in R} v_x = v_y\Big) \Rightarrow \Big(\bigwedge_{(x,y)\in R} f_x(v) = f_y(v)\Big)$$

is true for all $v \in \mathbb{R}^X$.

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Relates variables with identical ODE solutions when initialised equally

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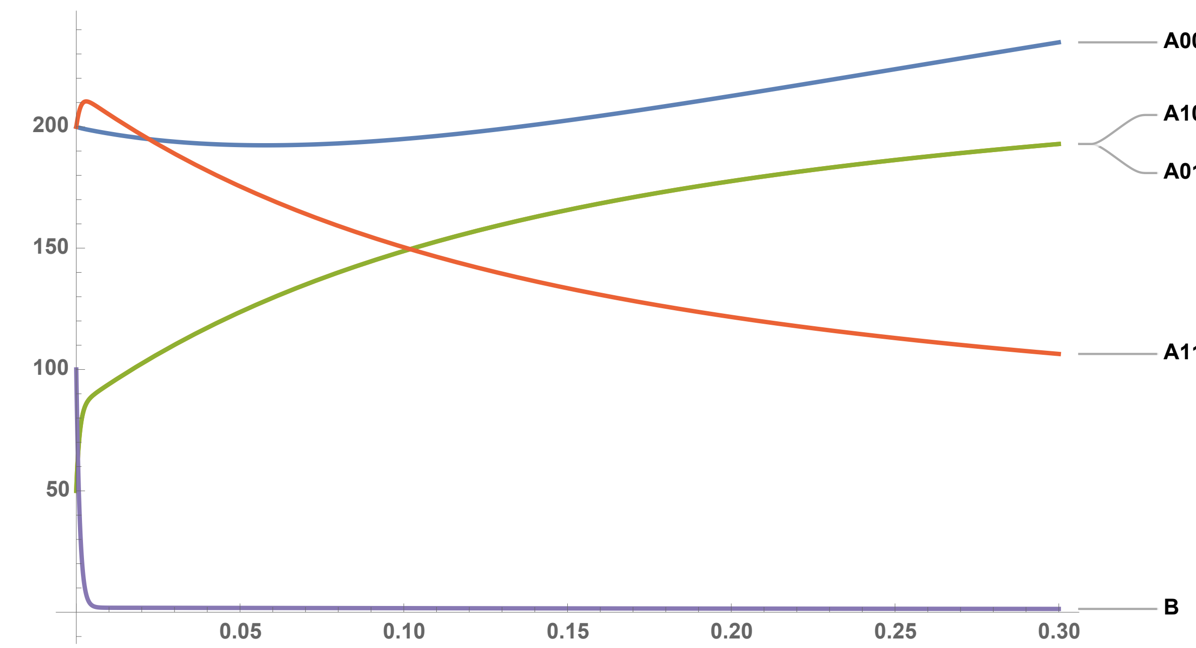

$$\dot{A}_{00} = -4A_{00}B + 3A_{10} + 3A_{01}$$
$$\dot{A}_{01} = 2A_{00}B - 3A_{01} - A_{01}B + 3A_{11}$$
$$\dot{A}_{10} = 2A_{00}B - 3A_{10} - A_{10}B + 3A_{11}$$
$$\dot{A}_{11} = A_{10}B + A_{01}B - 6A_{11}$$
$$\dot{B} = -4A_{00}B + 3A_{10} + 3A_{01} - A_{10}B - A_{01}B + 6A_{11}$$

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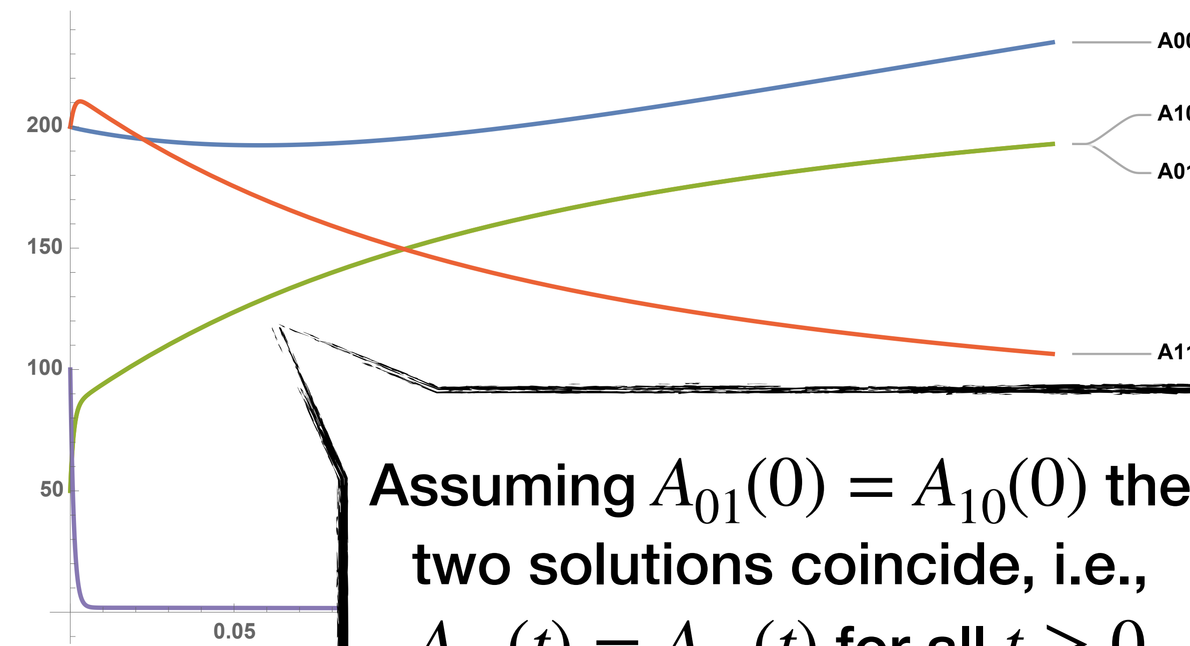

Assuming $A_{01}(0) = A_{10}(0)$ the two solutions coincide, i.e., $A_{01}(t) = A_{10}(t)$ for all $t \geq 0$

$$\dot{A}_{00} = -4A_{00}B + 3A_{10} + 3A_{01}$$
$$\dot{A}_{01} = 2A_{00}B - 3A_{01} - A_{01}B + 3A_{11}$$
$$\dot{A}_{10} = 2A_{00}B - 3A_{10} - A_{10}B + 3A_{11}$$
$$\dot{A}_{11} = A_{10}B + A_{01}B - 6A_{11}$$
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Proving Backward Equivalence

[Bacci[2], Larsen, Tribastone, Tschaikowski, Vandin'21]

We want to find some $R \subseteq X \times X$ satisfying

$$\forall v \in \mathbb{R}^X.\ \bigwedge_{(x,y)\in R} v_x = v_y \implies p(v) = q(v)$$

….and **explain why the implication holds**

Our solution

- We introduce a variant of **Strassen's theorem** for proving dominance between polynomial functions

- A witness of the implication is given via two type of couplings:
  (1) **Monomials couplings**: lift equivalence among variables to equivalence among monomials
  (2) **Linear couplings**: lift equivalence among variables to equivalence among linear functions

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Lift equivalence over variables to equivalences over polynomials

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Monomial & Linear couplings

Monomial Coupling

$\rho \in \Gamma_{\mathbf{M}}(m, n)$ **iff** $\rho : X \times X \to \mathbb{N}$ such that

- $\sum_{y \in X} \rho(x, y) = m(x)$ for all $x \in X$
- $\sum_{x \in X} \rho(x, y) = n(y)$ for all $y \in X$
- $\rho(x, y) \geq 0$ for all $x, y \in X$

$m = v^4 w^4$

$n = x^2 y^3 z^3$

	v^4	w^4
x^2		2
y^3	3	
z^3	1	2

Linear Coupling

$\omega \in \Gamma_{\mathbf{L}}(g, h)$ **iff** $\omega : X \times X \to \mathbb{R}$ such that

- $\sum_{y \in X} \omega(x, y) = (g^+ + h^-)(x)$ for all $x \in X$
- $\sum_{x \in X} \omega(x, y) = (h^+ + g^-)(y)$ for all $y \in X$
- $\omega(x, y) \geq 0$ for all $x, y \in X$

$g = 4v + 4w - 2x$

$h = 3y\ 3z$

	$4v$	$4w$
$2x$		2
$3y$	3	
$3z$	1	2

Monomial & Linear couplings

Lifting Relations

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$m = v^4 w^4$

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$$M[R] = \{(m, n) \mid \rho \in \Gamma_{\mathbf{M}}(m, n),\ supp(\rho) \subseteq R\}$$

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$3z$	1	2

$$L[R] = \{(g, h) \mid \omega \in \Gamma_{\mathbf{L}}(g, h),\ supp(\omega) \subseteq R\}$$

Lifting Metrics

Future work

Future work

Monomial & Linear couplings

Lifting Relations

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Lifting Metrics

Future work

Future work

Lifting Equality to Monomials

Consider the monomials m and n

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$$n = v^4 w^4$$

	v^4	w^4
x^2		
y^3		
z^3		

Lifting Equality to Monomials

Consider the monomials m and n

$$m = x^2 y^3 z^3$$

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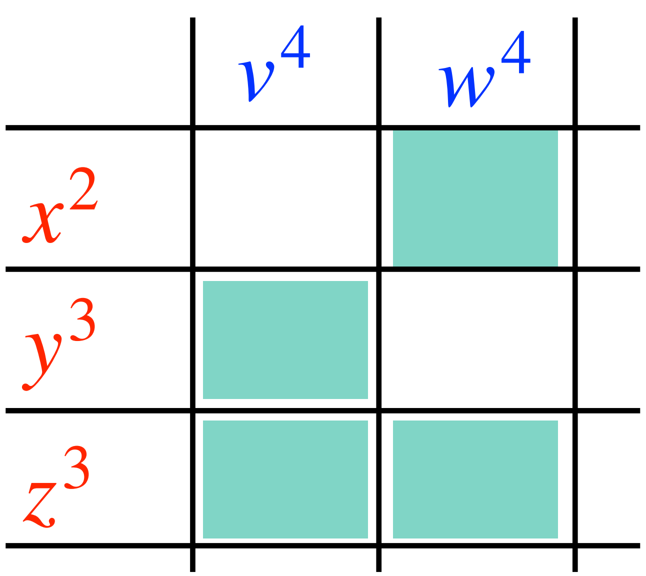
…assume that $x = w$, $y = v$, and $z = v = w$



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	v^4	w^4
x^2	0	2
y^3	3	0
z^3	1	2

$$\begin{aligned} m &= x^2 y^3 z^3 \\ &= (x^0 x^2)(y^3 y^0)(z^1 z^2) \\ &= (v^0 v^3 v^1)(w^2 w^0 w^2) \end{aligned}$$

$$n = v^4 w^4$$

Coupling Method for Polynomials

Linear Couplings

Theorem: Let $R \subseteq X \times X$ be an equivalence relation. The following are equivalent

(1) $(g, h) \in \mathbf{L}[R]$

(2) For all $v \in \mathbb{R}^X$, $\bigwedge_{(x,y)\in R} v_x \leq v_y \Rightarrow g(v) \leq h(v)$

(3) For all $v \in \mathbb{R}^X$, $\bigwedge_{(x,y)\in R} v_x = v_y \Rightarrow g(v) = h(v)$

Moreover $(1) \Rightarrow (2) \wedge (3)$ **hold for any relation** R

Monomial Couplings

Theorem: Let $R \subseteq X \times X$ be an equivalence relation. The following are equivalent

(1) $(m, n) \in \mathbf{M}[R]$

(2) For all $v \in \mathbb{R}^X_{>0}$, $\bigwedge_{(x,y)\in R} v_x \leq v_y \Rightarrow m(v) \leq n(v)$

(3) For all $v \in \mathbb{R}^X$, $\bigwedge_{(x,y)\in R} v_x = v_y \Rightarrow m(v) = n(v)$

Moreover $(1) \Rightarrow (2) \wedge (3)$ **holds for any relation** R

$$P[R] := L[M[R]]$$

Corollary: $(p, q) \in P[R]$ implies $\left(\bigwedge_{(x,y)\in R} v_x = v_y \Rightarrow p(v) = q(v)\right)$ for all $v \in \mathbb{R}^X$.

Backward Differential Bisimulation

We define **BDB as a post-fixed point** of the following operator.

$$\mathscr{B}(R) = \left\{(x, y) \mid (f_x, f_y) \in P[R]\right\}$$

From here we provided an <u>on-the-fly</u> procedure to test BDE which <u>exploits up-to techniques</u> (see Bacci et al. LICS'21])

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$$\begin{tikzcd} x \arrow[r, dash, "R"] \arrow[d, mapsto] & y \arrow[d, mapsto] \\ f_x \arrow[r, dash, "{P[R]}"'] & f_y \end{tikzcd}$$

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Theorem:
For any $R \subseteq X \times X$

(1) If R is a BDB, then R^* is a BDE

(2) If R is a BDE, then R is a BDB

(3) gfp$(\mathscr{B})$ is the greatest BDE

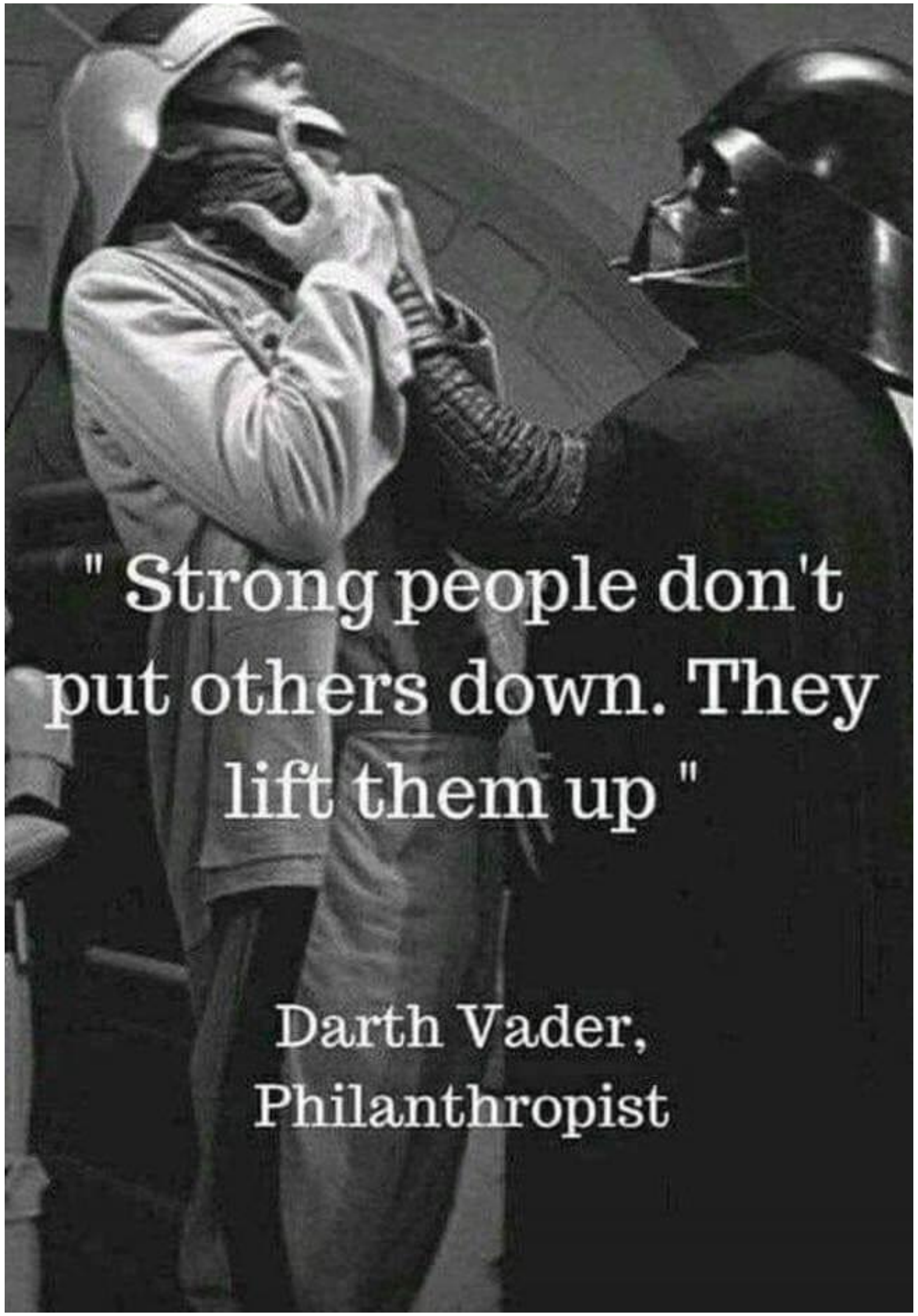
From here we provided an on-the-fly procedure to test BDE which exploits up-to techniques (see Bacci et al. LICS’21])

Almost done

…stay awake a few more seconds

Conclusion

- Comparing the behaviours reduces to lifting relations/distances from states to other structures
- Coupling is a powerful technique to
  - Define new behavioural equivalences & metrics
  - Define algorithms to compute them
  - Prove useful properties
    - Approximate minimisation
    - LTL dissimilarity upper-bounds



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