Overview
Examples: while loop, Markov chain

State: $X = (p_{bull}, p_{bear}, p_{stag}) \in [0, 1]^3$

Transitions:

$A = \begin{bmatrix}
0.9 & 0.075 & 0.025 \\
0.15 & 0.8 & 0.25 \\
0.25 & 0.05 & 0.25
\end{bmatrix}$

Linear dynamical system

$X_{n+1} = AX_n$

Linear loop

$p_{bull} := 0$

$p_{bear} := 1$

$p_{stag} := 0$

while $p_{bull} \leq 1/2$ do

$\begin{bmatrix}
p_{bull} \\
p_{bear} \\
p_{stag}
\end{bmatrix} := A \begin{bmatrix}
p_{bull} \\
p_{bear} \\
p_{stag}
\end{bmatrix}$

The loop terminates if and only if the probability of a bull market is $> 1/2$. 
Examples: while loop, Markov chain

**State:** \( X = (p_{bull}, p_{bear}, p_{stag}) \in [0, 1]^3 \)

**Transitions:**

\[
A = \begin{bmatrix}
0.9 & 0.15 & 0.25 \\
0.075 & 0.8 & 0.25 \\
0.025 & 0.05 & 0.5
\end{bmatrix}
\]

→ Linear dynamical system

\[ X_{n+1} = AX_n \]
Examples: while loop, Markov chain

State: \( X = (p_{bull}, p_{bear}, p_{stag}) \in [0, 1]^3 \)

Transitions:

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A = \begin{bmatrix}
0.9 & 0.15 & 0.25 \\
0.075 & 0.8 & 0.25 \\
0.025 & 0.05 & 0.5 \\
\end{bmatrix}
\]

→ Linear dynamical system

\[
x_{n+1} = Ax_n
\]
State: $X = (p_{\text{bull}}, p_{\text{bear}}, p_{\text{stag}}) \in [0, 1]^3$

Transitions:

$$A = \begin{bmatrix} 0.9 & 0.15 & 0.25 \\ 0.075 & 0.8 & 0.25 \\ 0.025 & 0.05 & 0.5 \end{bmatrix}$$

$\rightarrow$ Linear dynamical system

$$X_{n+1} = AX_n$$

Linear loop

$$p_{\text{bull}} := 0 \quad p_{\text{bear}} := 1 \quad p_{\text{stag}} := 0$$

while $p_{\text{bull}} \leq 1/2$ do

$$\begin{bmatrix} p_{\text{bull}} \\ p_{\text{bear}} \\ p_{\text{stag}} \end{bmatrix} := A \begin{bmatrix} p_{\text{bull}} \\ p_{\text{bear}} \\ p_{\text{stag}} \end{bmatrix}$$

The loop terminates if and only if the probability of a bull market is $> 1/2$. 
Example: mass-spring-damper system

\[ \begin{aligned}
  \text{Equation of motion:} & \quad m \ddot{z} = -kz - b \dot{z} + mg + u \\
  \text{State:} & \quad X = (z, \dot{z}, 1) \in \mathbb{R}^3
\end{aligned} \]
Example: mass-spring-damper system

State: $X = z \in \mathbb{R}$

Equation of motion:

$$mz'' = -kz - bz' + mg$$
Example: mass-spring-damper system

State: \( X = z \in \mathbb{R} \)

Equation of motion:

\[
mz'' = -kz - bz' + mg
\]

→ Affine but not first order
Example: mass-spring-damper system

State: \( X = z \in \mathbb{R} \)

Equation of motion:
\[
mz'' = -kz - bz' + mg
\]
→ Affine but not first order

State: \( X = (z, z', 1) \in \mathbb{R}^3 \)

Equation of motion:
\[
\begin{bmatrix} z' \\ z' \\ 1 \end{bmatrix}' = \begin{bmatrix} -\frac{k}{m}z - \frac{b}{m}z' + g \\ 0 \end{bmatrix}
\]
Can be used to model a car suspension.
Example: mass-spring-damper system

\[ m \ddot{z} = -kz - bz' + mg \]

→ Affine but not first order

State: \( X = (z, z', 1) \in \mathbb{R}^3 \)

Equation of motion:

\[
\begin{bmatrix}
  z' \\
  z'' \\
  1
\end{bmatrix} = 
\begin{bmatrix}
  -\frac{k}{m}z - \frac{b}{m}z' + g \\
  0
\end{bmatrix}
\]

→ Linear dynamical system

\[ X' = AX \]
Example: mass-spring-damper system

\[ m \frac{d^2 z}{dt^2} = -kz - bz' + mg + u \]

State: \( X = z \in \mathbb{R} \)

Equation of motion:

with external input \( u(t) \).

State: \( X = (z, z', 1) \in \mathbb{R}^3 \)

Equation of motion:

\[
\begin{bmatrix}
    z' \\
    z''
\end{bmatrix} = \begin{bmatrix}
    -k & z' \\
    -\frac{b}{m} & z' \\
    0 & g
\end{bmatrix}
\]

Can be used to model a car suspension.
Example: mass-spring-damper system

![Diagram of a mass-spring-damper system](image)

with external input $u(t)$.

→ Linear time invariant system

$$X' = AX + Bu$$

State: $X = z \in \mathbb{R}$

Equation of motion:

$$mz'' = -kz - bz' + mg + u$$

State: $X = (z, z', 1) \in \mathbb{R}^3$

Equation of motion:

$$\begin{bmatrix} z' \\ z'' \\ 1 \end{bmatrix} = \begin{bmatrix} -\frac{k}{m}z - \frac{b}{m}z' + g \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \frac{1}{m} \end{bmatrix} u$$
Example: mass-spring-damper system

\[ mz'' = -kz - bz' + mg + u \]

State: \( X = z \in \mathbb{R} \)

Equation of motion:

State: \( X = (z, z', 1) \in \mathbb{R}^3 \)

Equation of motion:

Can be used to model a car suspension.
Linear dynamical systems

Discrete case

\[ x(n + 1) = Ax(n) \]

- biology,
- software verification,
- probabilistic model checking,
- combinatorics,
- ...

Continuous case

\[ x'(t) = Ax(t) \]

- biology,
- physics,
- probabilistic model checking,
- electrical circuits,
- ...

Typical questions

- reachability
- safety
Linear dynamical systems

Discrete case

\[ x(n + 1) = Ax(n) + Bu(n) \]

- biology,
- software verification,
- probabilistic model checking,
- combinatorics,
- ....

Continuous case

\[ x'(t) = Ax(t) + Bu(t) \]

- biology,
- physics,
- probabilistic model checking,
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- ....

Typical questions

- reachability
- safety
- controllability
Linear dynamical systems

Discrete case

\[ x(n + 1) = Ax(n) + Bu(n) \]

- biology,
- software verification,
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- combinatorics,
- ....

Continuous case

\[ x'(t) = Ax(t) + Bu(t) \]

- biology,
- physics,
- probabilistic model checking,
- electrical circuits,
- ....

Typical questions

- reachability
- safety
- controllability
- optimal control
- feedback control
- ...

5 / 101
More complicated programs

**Linear loop with if**

\[
\begin{align*}
x & := 2^{-10} \\
y & := 1 \\
\text{while } y \geq x \text{ do} \\
& \quad \text{if } y \geq 2x \text{ then} \\
& \quad \quad \begin{bmatrix} x \\ y \end{bmatrix} := \begin{bmatrix} 2 & 0 \\ 1 & 4 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} \\
& \quad \text{else} \\
& \quad \quad \begin{bmatrix} x \\ y \end{bmatrix} := \begin{bmatrix} 2 & 3 \\ -3 & 7 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}
\end{align*}
\]
More complicated programs

Linear loop with if

\[
\begin{align*}
  x &:= 2^{-10} \\
  y &:= 1 \\
\text{while } y \geq x \text{ do} & \\
  \quad \text{if } y \geq 2x \text{ then} & \\
  \quad \quad \begin{bmatrix} x \\ y \end{bmatrix} := \begin{bmatrix} 2 & 0 \\ 1 & 4 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} \\
  \quad \text{else} & \\
  \quad \quad \begin{bmatrix} x \\ y \end{bmatrix} := \begin{bmatrix} 2 & 3 \\ -3 & 7 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}
\end{align*}
\]

Very challenging to analyze!

- reachability is undecidable
- invariant* synthesis also hard

*Will be defined later, think “approximate reachability”.
More complicated programs

**Linear loop with if**

\[ x := 2^{-10} \]
\[ y := 1 \]
while \( y \geq x \) do
  if \( y \geq 2x \) then
    \[
    \begin{bmatrix}
    x \\
    y
    \end{bmatrix}
    :=
    \begin{bmatrix}
    2 & 0 \\
    1 & 4
    \end{bmatrix}
    \begin{bmatrix}
    x \\
    y
    \end{bmatrix}
    \]
  else
    \[
    \begin{bmatrix}
    x \\
    y
    \end{bmatrix}
    :=
    \begin{bmatrix}
    2 & 3 \\
    -3 & 7
    \end{bmatrix}
    \begin{bmatrix}
    x \\
    y
    \end{bmatrix}
    \]

**Nondeterministic loop**

\[ x := 2^{-10} \]
\[ y := 1 \]
while true do
  non deterministically do
    \[
    \begin{bmatrix}
    x \\
    y
    \end{bmatrix}
    :=
    \begin{bmatrix}
    2 & 0 \\
    1 & 4
    \end{bmatrix}
    \begin{bmatrix}
    x \\
    y
    \end{bmatrix}
    \]
  or
    \[
    \begin{bmatrix}
    x \\
    y
    \end{bmatrix}
    :=
    \begin{bmatrix}
    2 & 3 \\
    -3 & 7
    \end{bmatrix}
    \begin{bmatrix}
    x \\
    y
    \end{bmatrix}
    \]

Very challenging to analyze!
- reachability is undecidable
- invariant\(^*\) synthesis also hard

\(^*\)Will be defined later, think “approximate reachability”.
More complicated programs

Linear loop with if

\[
\begin{align*}
x &:= 2^{-10} \\
y &:= 1 \\
\text{while } y \geq x \text{ do} \\
\quad &\text{if } y \geq 2x \text{ then} \\
\quad &\quad \begin{bmatrix} x \\ y \end{bmatrix} := \begin{bmatrix} 2 & 0 \\ 1 & 4 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} \\
\quad &\text{else} \\
\quad &\quad \begin{bmatrix} x \\ y \end{bmatrix} := \begin{bmatrix} 2 & 3 \\ -3 & 7 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}
\end{align*}
\]

Nondeterminic loop

\[
\begin{align*}
x &:= 2^{-10} \\
y &:= 1 \\
\text{while true do} \\
\quad &\text{non deterministically do} \\
\quad &\quad \begin{bmatrix} x \\ y \end{bmatrix} := \begin{bmatrix} 2 & 0 \\ 1 & 4 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} \\
\quad &\text{or} \\
\quad &\quad \begin{bmatrix} x \\ y \end{bmatrix} := \begin{bmatrix} 2 & 3 \\ -3 & 7 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}
\end{align*}
\]

Very challenging to analyze!

- reachability is undecidable
- invariant* synthesis also hard

Overapproximate behaviours

- reachability still undecidable
- invariant synthesis possible

*Will be defined later, think “approximate reachability”.
Affine program

\[
\begin{align*}
x &:= 2^{-10} \\
y &:= 1 \\
\text{while } y \geq x \text{ do} \\
\begin{bmatrix} x \\ y \end{bmatrix} &:= \begin{bmatrix} 2 & 0 \\ \frac{7}{4} & \frac{1}{4} \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}
\end{align*}
\]
Does this program halt?

Affine program

\[
x := 2^{-10} \\
y := 1 \\
\text{while } y \geq x \text{ do} \\
\begin{bmatrix} x \\ y \end{bmatrix} := \begin{bmatrix} 2 & 0 \\ \frac{7}{4} & \frac{1}{4} \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}
\]

Certificate of non-termination:

\[
x^2 - x^3 = 1023 - 1073741824
\]
Does this program halt?

Affine program

\[
\begin{align*}
x & := 2^{-10} \\
y & := 1 \\
\text{while } y \geq x \text{ do} \\
\begin{bmatrix} x \\ y \end{bmatrix} & := \begin{bmatrix} 2 & 0 \\ \frac{7}{4} & \frac{1}{4} \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}
\end{align*}
\]

Certificate of non-termination:

\[
x^2 - x^3 = 1023 - 1073741824
\]
Does this program halt?

Affine program

\[
x := 2^{-10} \\
y := 1 \\
\text{while } y \geq x \text{ do} \\
\begin{bmatrix} x \\ y \end{bmatrix} := \begin{bmatrix} 2 & 0 \\ \frac{7}{4} & \frac{1}{4} \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}
\]

Certificate of non-termination:

\[x^2y - x^3 = 1023 \quad 1073741824 \tag{1}\]

\(\begin{bmatrix} x \\ y \end{bmatrix} \Rightarrow \begin{bmatrix} 2 \\ 7 \end{bmatrix}\) is an invariant: it holds at every step.
Does this program halt?

Affine program

\[ x := 2^{-10} \]
\[ y := 1 \]
while \( y \geq x \) do
\[
\begin{bmatrix} x \\ y \end{bmatrix} := \begin{bmatrix} 2 & 0 \\ \frac{7}{4} & \frac{1}{4} \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}
\]

Certificate of non-termination:
\[ x^2 - x^3 = 1023 - 1073741824 \]
Does this program halt?

Affine program

\[
\begin{align*}
x &:= 2^{-10} \\
y &:= 1 \\
\text{while } y \geq x \text{ do} \\
\begin{bmatrix} x \\ y \end{bmatrix} &:= 
\begin{bmatrix} 2 & 0 \\ \frac{7}{4} & \frac{1}{4} \end{bmatrix} 
\begin{bmatrix} x \\ y \end{bmatrix}
\end{align*}
\]

Certificate of non-termination:

\[
x^2 y - x^3 = 1023 - 1073741824 (1)
\]

\[\begin{bmatrix} x \\ y \end{bmatrix} \overset{\text{(2)}}{\implies} \text{guard is true} \]

\[
7/101
\]
Does this program halt?

**Affine program**

\[
\begin{align*}
x & := 2^{-10} \\
y & := 1 \\
\text{while } y \geq x \text{ do} \\
\begin{bmatrix} x \\ y \end{bmatrix} & := \begin{bmatrix} 2 & 0 \\ \frac{7}{4} & \frac{1}{4} \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}
\end{align*}
\]

Certificate of non-termination:

\[
x^2 y - x^3 = \frac{1023}{1073741824}
\]
Does this program halt?

Affine program

\[
x := 2^{-10} \\
y := 1 \\
\text{while } y \geq x \text{ do} \\
\begin{bmatrix} x \\ y \end{bmatrix} := \begin{bmatrix} 2 & 0 \\ \frac{7}{4} & \frac{1}{4} \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}
\]

Certificate of non-termination:

\[
x^2 y - x^3 = \frac{1023}{1073741824}
\] (1)

(2) is an invariant: it holds at every step
Does this program halt?

Affine program

\[
\begin{align*}
x &:= 2^{-10} \\
y &:= 1 \\
\text{while } y \geq x \text{ do} \\
\begin{bmatrix} x \\
y \end{bmatrix} &:= \begin{bmatrix} 2 & 0 \\
7/4 & 1/4 \end{bmatrix} \begin{bmatrix} x \\
y \end{bmatrix}
\end{align*}
\]

Certificate of non-termination:

\[
x^2 y - x^3 = \frac{1023}{1073741824}
\]

- (2) is an invariant: it holds at every step
- (2) implies the guard is true
invariant = overapproximation of the reachable states
Invariants

\[
\text{invariant} = \text{overapproximation of the reachable states}
\]

\textbf{inductive} invar\textit{i}ant = invar\textit{i}ant \textit{preserved by the transition relation}
The classical approach to the verification of temporal safety properties of programs requires the construction of inductive invariants [...]. Automation of this construction is the main challenge in program verification.

D. Beyer, T. Henzinger, R. Majumdar, and A. Rybalchenko
Invariant Synthesis for Combined Theories, 2007
Affine programs

- Nondeterministic branching (no guards)
- All assignments are affine
- Allow nondeterministic assignments (\(x := \ast\))
- Can overapproximate complex programs
- Covers existing formalisms: finite, probabilistic, quantum, quantitative automata
Affine programs

- Nondeterministic branching (no guards)

![Diagram of an affline program with nodes 1, 2, and 3 connected by edges labeled $f_1$, $f_2$, $f_3$, $f_4$, and $f_5$.]
Affine programs

- Nondeterministic branching (no guards)
- All assignments are affine

\[
x := 3x - 7y + 1
\]
Affine programs

- Nondeterministic branching (no guards)
- All assignments are affine
- Allow nondeterministic assignments ($x := *$)

$x := 3x - 7y + 1$

Diagram:

1. $x := 3x - 7y + 1$
2. $f_2$
3. $f_5$
4. $f_3$
5. $y := *$
Affine programs

- Nondeterministic branching (no guards)
- All assignments are affine
- Allow nondeterministic assignments ($x := *$)

$x := 3x - 7y + 1$

- Can overapproximate complex programs
- Covers existing formalisms: finite, probabilistic, quantum, quantitative automata
**RC circuit**

\[
\begin{align*}
\dot{I} &= 0 \\
\dot{I}_R &= -\frac{1}{RC}I \\
\dot{V}_R &= -\frac{1}{C}I \\
\dot{Q} &= I \\
\dot{V}_C &= \frac{1}{C}I
\end{align*}
\]
RC circuit

**OPEN**

\[
\begin{align*}
\dot{I} &= 0 \\
\dot{I}_R &= -\frac{1}{RC} I_R \\
\dot{V}_R &= -\frac{1}{C} I_R \\
\dot{Q} &= I_R \\
\dot{V}_C &= \frac{1}{C} I_R 
\end{align*}
\]
**RC circuit**

### OPEN

\[
\begin{align*}
\dot{I} &= 0 \\
\dot{I}_R &= -\frac{1}{RC} I_R \\
\dot{V}_R &= -\frac{1}{C} I_R \\
\dot{Q} &= I_R \\
\dot{V}_C &= \frac{1}{C} I_R
\end{align*}
\]

### CLOSED

\[
\begin{align*}
\dot{I} &= -\frac{1}{RC} I_R \\
\dot{I}_R &= -\frac{1}{RC} I_R \\
\dot{V}_R &= -\frac{1}{C} I_R \\
\dot{Q} &= I_R \\
\dot{V}_C &= \frac{1}{C} I_R
\end{align*}
\]
RC circuit

\[ \dot{I} = \frac{1}{R}(V - V_C) \]

\[ I = \frac{1}{R}(V - V_C) \]

\[ I_R = \frac{1}{RC} I \]

\[ Q = I_R \]

\[ V_C = \frac{1}{C} I_R \]

\[ \dot{V} \]

\[ \dot{I} \]

\[ \dot{Q} = I_R \]

\[ V_C = \frac{1}{C} I_R \]
Switching systems

Restricted hybrid system:

- linear dynamics
- no guards (nondeterministic)
- no discrete updates

Switching process:

\[ x' = A_1 x \quad \Rightarrow \quad x' = A_2 x \quad \Rightarrow \quad x' = A_3 x \quad \Rightarrow \quad x' = A_4 x \]

Reachability is also undecidable.

Invariant synthesis is possible.
Switching systems

Restricted hybrid system:
- linear dynamics
- no guards (nondeterministic)
- no discrete updates

Reachability also undecidable

Invariant synthesis possible
Going hybrid: a bouncing ball

\[ \begin{align*}
\dot{x} &= v_x \\
\dot{y} &= v_y \\
\dot{v}_x &= 0 \\
\dot{v}_y &= -g \\
t &= 0 \\
x &= 0 \\
y &= h \\
v_x &= c \\
v_y &= 0
\end{align*} \]

Invariants:

\[ \begin{align*}
v_x &= c \\
x &= tc \\
v_y^2 + 2g(y - h) &= 0
\end{align*} \]

recover conservation of energy!
Going hybrid: a bouncing ball

\[ \begin{align*}
\dot{x} &= v_x \\
\dot{y} &= v_y \\
\dot{v}_x &= 0 \\
\dot{v}_y &= -g \\
\end{align*} \]

\[ t = 1 \]

\[ x = 0 \]

\[ y = h \]

\[ v_x := c \]

\[ v_y := 0 \]

\[ v_y := -v_y \]

▷ affine program: collision

+ linear differential equation: mechanics

= linear hybrid automaton

recover conservation of energy!
Going hybrid: a bouncing ball

\[ \dot{x} = v_x \]
\[ \dot{y} = v_y \]
\[ \dot{v}_x = 0 \]
\[ \dot{v}_y = -g \]
\[ \dot{t} = 1 \]

\[ v_y := -v_y \]

- affine program: collision
- linear differential equation: mechanics
- linear hybrid automaton

Invariants:
- \( v_x = c \)
- \( x = tc \)
- \( v_y^2 + 2g(y - h) = 0 \)

recover conservation of energy!
Linear Hybrid Automata

- Nondeterministic branching (no guards)
- All assignments are affine
- **Linear differential equations** in each location

\[
\begin{align*}
\dot{x} &= 2y - x \\
\dot{y} &= x - y \\
x &:= 3x - 7y + 1 \\
\dot{X} &= AX \\
\dot{X} &= BX
\end{align*}
\]
Linear Hybrid Automata

- Nondeterministic branching (no guards)
- All assignments are affine
- **Linear differential equations** in each location

\[ \begin{align*}
\dot{x} &= 2y - x \\
\dot{y} &= x - y \\
x &= 3x - 7y + 1 \\
\dot{X} &= AX \\
\dot{X} &= BX
\end{align*} \]

- More general than affine programs
- More general than linear differential equations
Which invariants?

Octagons \(\subseteq\) Polyhedrons \(\subseteq\) Semialgebraic sets

Intervals \(\supseteq\) Affine/linear sets \(\supseteq\) Algebraic sets = polynomial equalities
Linear system with rounding

Rounding: $\lfloor . \rceil = \text{round to nearest integer}$

$A = \begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix} \in \mathbb{Q}^{2 \times 2}$,

$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \lfloor x \rceil \\ \lfloor y \rceil \end{pmatrix}$

Problem: given $X_0 \in \mathbb{Q}^{2 \times 2}$, define $X_{n+1} = \lfloor AX_n \rceil$.

- Is reachability decidable?
- Is $(X_n)_{n}$ eventually periodic?
- What does the reachable set look like?

Open problems! Only known for a few specific values of $\theta$. 

$r = 10$, $\theta = \pi/42$

$r = 10$, $\theta = 20.4\pi$

$r = 15$, $\theta = \pi/91$

$r = 20$, $\theta = \pi/14$
Linear system with rounding

Rounding: \([\cdot] = \text{round to nearest integer}\)

\[ A = \begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix} \in \mathbb{Q}^{2 \times 2}, \quad \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \lfloor x \rfloor \\ \lfloor y \rfloor \end{pmatrix} \]

Problem: given \(X_0 \in \mathbb{Q}^2\), define \(X_{n+1} = [AX_n]\)

- is reachability decidable?
- is \((X_n)_n\) eventually periodic?
- what does the reachable set look like?

Open problems! Only known for a few specific values of \(\theta\).
Linear system with rounding

Rounding: $\lfloor \cdot \rceil = \text{round to nearest integer}$

$$A = \begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix} \in \mathbb{Q}^{2 \times 2}, \quad \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \lfloor x \rfloor \\ \lfloor y \rfloor \end{pmatrix}$$

Problem: given $X_0 \in \mathbb{Q}^2$, define $X_{n+1} = \lfloor AX_n \rceil$

- is reachability decidable?
- is $(X_n)_n$ eventually periodic?

what does the reachable set look like?

$r = 10, \theta = \pi/42$
Linear system with rounding

Rounding: $\lfloor \cdot \rceil = \text{round to nearest integer}$

$$A = \begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix} \in \mathbb{Q}^{2 \times 2}, \quad \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \lfloor x \rceil \\ \lfloor y \rceil \end{pmatrix}$$

Problem: given $X_0 \in \mathbb{Q}^2$, define $X_{n+1} = \lfloor AX_n \rceil$

- is reachability decidable?
- is $(X_n)_n$ eventually periodic?

what does the reachable set look like?

$r = 10, \theta = \pi/42 \quad r = 10, \theta = \frac{2^{0.4\pi}}{10}$
Linear system with rounding

**Rounding:** \([\cdot]\) = round to nearest integer

\[ A = \begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix} \in \mathbb{Q}^{2 \times 2}, \quad \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \lfloor x \rfloor \\ \lfloor y \rfloor \end{pmatrix} \]

**Problem:** given \( X_0 \in \mathbb{Q}^2 \), define \( X_{n+1} = [AX_n] \)

- is reachability decidable?
- is \((X_n)_n\) eventually periodic?

- what does the reachable set look like?

\( r = 10, \theta = \pi/42 \) \quad \( r = 10, \theta = \frac{2^{0.4} \pi}{10} \) \quad \( r = 15, \theta = \pi/91 \)
Linear system with rounding

Rounding: \( \lfloor \cdot \rfloor \) = round to nearest integer

\[
A = \begin{bmatrix}
\cos \theta & -\sin \theta \\
\sin \theta & \cos \theta
\end{bmatrix} \in \mathbb{Q}^{2 \times 2},
\begin{pmatrix}
x \\
y
\end{pmatrix} = \begin{pmatrix}
\lfloor x \rfloor \\
\lfloor y \rfloor
\end{pmatrix}
\]

Problem: given \( X_0 \in \mathbb{Q}^2 \), define \( X_{n+1} = [AX_n] \)

- is reachability decidable?
- is \((X_n)_n\) eventually periodic?
- what does the reachable set look like?

\[
\begin{align*}
r &= 10, \quad \theta = \pi/42 & r &= 10, \quad \theta = \frac{2^{0.4}\pi}{10} & r &= 15, \quad \theta = \pi/91 & r &= 20, \quad \theta = \pi/14
\end{align*}
\]
Linear system with rounding

Rounding: \(\lfloor \cdot \rceil = \text{round to nearest integer}\)

\[
A = \begin{bmatrix}
\cos \theta & -\sin \theta \\
\sin \theta & \cos \theta \\
\end{bmatrix} \in \mathbb{Q}^{2 \times 2}, \quad \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \lfloor x \rceil \\ \lfloor y \rceil \end{pmatrix}
\]

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- is reachability decidable?
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- what does the reachable set look like?

Open problems! Only known for a few specific values of \(\theta\).
Linear dynamical systems are ubiquitous...

... and lead to very interesting mathematics!
Interesting related mathematics

- **Linear recurrent sequences (LRS)**

  \[ x_{n+k} = a_{k-1}a_{n+k-1} + \cdots + x_0x_n \]

  **Fibonacci:** \( F_{n+2} = F_{n+1} + F_n \)

- **Skolem/Positivity problem (Open for more than 70 years!)**
  - Decide if a given LRS has a zero/is always positive

- **Exponential polynomials:**

  \[ f(t) = P_1(t)e^{\lambda_1 t} + \cdots + P_n(t)e^{\lambda_n t} \]

  Examples: polynomials, \( e^t \), \( \sin(t) \), \( t^2\sin(t) - e^{-t} \)

- **Continuous Skolem/Positivity (Also open)**
  - Decide if an exponential polynomial has a zero/is always positive

  Reachability often harder/reduces to of these problems!
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- Continuous Skolem/Positivity (Also open)
  decide if an exponential polynomial has a zero/is always positive
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Algebraic numbers and conjectures

**Algebraic number**: root of polynomial with integer coefficients

**Transcendental number**: not algebraic, e.g. $e$, $\pi$

Theorem (Gelfond–Schneider theorem)

If $a$, $b$ are algebraic numbers with $a \neq 0$, $1$ and $b$ irrational, then (any value of) $a^b$ transcendental.

Example: $2^\sqrt{2}$ is transcendental.

Why is this related to reachability?

- Target is usually rational/algebraic
- Reachability creates constraints between numbers

Example: given $a$, $b \in \mathbb{Q}$, $P \in \mathbb{Q}[X]$ polynomial, find $t$ such that $P(t) = a$ and $e^t = b$; impossible unless $t = 0$.

Biggest open question in this field: Schanuel's conjecture
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Transcendental number theory

Many problems boil down to **diophantine equations/approximations**:
- Finding integer points in cones: Kronecker’s theorem

\[
\sqrt{2} + \log \sqrt{3} - 3 \log \sqrt{7} = 1 + \log 9 - \log 42 \sqrt{666}
\]
Many problems boil down to diophantine equations/approximations:

- Finding integer points in cones: Kronecker’s theorem

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Finitely generated matrix semigroup:
\( A_1, \ldots, A_k \in \mathbb{Q}^{n \times n} \) generate a semigroup \( S = \langle A_1, \ldots, A_k \rangle \)

Example: \( SL_2(\mathbb{Z}) = \left\langle \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & -1 \\ 1 & 1 \end{bmatrix} \right\rangle \)
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Problems:
- finiteness: is $S$ finite?
- mortality: does $0 \in S$?
- identity: does $I_n \in S$?
- membership: does $M \in S$ where $M \in \mathbb{Q}^{n \times n}$ is given as input?
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Undecidable in general, many decidable subclasses are known. Equivalent to reachability of affine programs.
Algebraic geometry

Study systems of multivariate polynomial equations using abstract algebraic techniques, with applications to geometry.

Examples

\[ x^2 + y^2 + z^2 - 1 = 0 \quad \leadsto \quad \text{sphere in } \mathbb{R}^3 \]

\[ x^2 + y^2 + z^2 = 1 \land x + y + z = 1 \quad \leadsto \quad \text{“sliced” sphere in } \mathbb{R}^3 \]

\[ x^2 + 1 = 0 \quad \leadsto \quad \emptyset \text{ in } \mathbb{R} \]

\[ x^2 + 1 = 0 \quad \leadsto \quad \{i, -i\} \text{ in } \mathbb{C} \]

The field \( K \) is very important:

▶ real algebraic geometry: more “intuitive” but more difficult, really requires the study of semi-algebraic sets

▶ mainstream algebraic geometry: \( K \) is algebraically closed \( \dagger \), e.g. \( \mathbb{C} \)

\( K \) is algebraically closed if every non-constant polynomial has a root in \( K \).
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The field \( \mathbb{K} \) is very important:

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First-order theory of the reals

Many questions expressible in first-order logical theories:

- $\mathfrak{R}_0 = (\mathbb{R}, 0, 1, <, +, \cdot)$: decidable

  \[ \forall x, y \in \mathbb{R} \quad \frac{x + y}{2} \geq \sqrt{xy} \]
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- $\mathcal{R}_{\text{exp}} = (\mathbb{R}, 0, 1, <, +, \cdot, \exp, \cos \upharpoonright_{[0,1]})$: decidable subject to Schanuel’s conjecture
  \[
  \forall x \in \mathbb{R} \quad x \neq 0 \Rightarrow t + te^t - 43e^{3t} \neq 1
  \]
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  \[
  \forall x \in \mathbb{R} \quad x \neq 0 \Rightarrow t + te^t - 43e^{3t} \neq 1
  \]

- Presburger arithmetic ($\mathbb{N}, 0, 1, <, +$): decidable
  \[
  \exists x \in \mathbb{N}^n \quad Ax \geq b
  \]
Linear dynamical systems are ubiquitous and exact reachability questions lead to very interesting mathematical and logical questions.
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But...

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  trees, arrays, ...
Summary

Linear dynamical systems are ubiquitous and exact reachability questions lead to very interesting mathematical and logical questions.

But...

- some systems are fundamentally nonlinear
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- some programs are not sequential / nondeterministic
  - probabilistic, concurrent/parallel, ...

Linear dynamical systems are ubiquitous and exact reachability questions lead to very interesting mathematical and logical questions.

But...

- some systems are fundamentally nonlinear
  \[ x_{n+1} = x_n^2 \]
- real programs manipulate data structures:
  trees, arrays, ...
- some programs are not sequential / nondeterministic
  probabilistic, concurrent/parallel, ...
- exact reachability is not the only approach
  testing, probabilistic model checking, incomplete algorithms, ...
Reachability
Examples: while loop, Markov chain

State: \( X = (p_{\text{bull}}, p_{\text{bear}}, p_{\text{stag}}) \in [0, 1]^3 \)

Transitions:

\[
A = \begin{bmatrix}
0.9 & 0.15 & 0.25 \\
0.075 & 0.8 & 0.25 \\
0.025 & 0.05 & 0.5 \\
\end{bmatrix}
\]

→ Linear dynamical system

\[ X_{n+1} = AX_n \]

Linear loop

\[
\begin{align*}
   p_{\text{bull}} & := 0 \\
   p_{\text{bear}} & := 1 \\
   p_{\text{stag}} & := 0
\end{align*}
\]

while \( p_{\text{bull}} \leq 1/2 \) do

\[
\begin{bmatrix}
   p_{\text{bull}} \\
   p_{\text{bear}} \\
   p_{\text{stag}}
\end{bmatrix}
\] := \( A \begin{bmatrix}
   p_{\text{bull}} \\
   p_{\text{bear}} \\
   p_{\text{stag}}
\end{bmatrix}\]

The loop terminates if and only if the probability of a bull market is \( > 1/2 \).
Does this loop terminate?

**Linear Loop**

\[ x := 2^{-10}, \quad y := 1 \]

**until** \( \phi(x) \) **do**

\[
\begin{bmatrix}
  x \\
  y
\end{bmatrix} :=
\begin{bmatrix}
  2 & 0 \\
  7/4 & 1/4
\end{bmatrix}
\begin{bmatrix}
  x \\
  y
\end{bmatrix}
\]
Termination Linear Loops

Does this loop terminate?

Linear Loop

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  \frac{7}{4} & \frac{1}{4}
\end{bmatrix}
\begin{bmatrix}
  x \\
y
\end{bmatrix}
\]

Reachability problem

Given

- initial point: \( x_0 \in \mathbb{Q}^d \),
- transition matrix: \( A \in \mathbb{Q}^{d \times d} \),
- target set: \( S \subseteq \mathbb{R}^d \)

decide if \( \exists n \in \mathbb{N}. \ A^n x_0 \in S \).
Termination Linear Loops

Does this loop terminate?

**Linear Loop**

\[
x := 2^{-10}, \quad y := 1
\]

until \(x = 42\) and \(y = 36\) do

\[
\begin{bmatrix}
x \\
y
\end{bmatrix} := \begin{bmatrix}
2 & 0 \\
\frac{7}{4} & \frac{1}{4}
\end{bmatrix} \begin{bmatrix}
x \\
y
\end{bmatrix}
\]

Natural choices for \(S\):

- **point:**
  \[
  \exists n \in \mathbb{N} \quad A^n x_0 = y
  \]
Termination Linear Loops

Does this loop terminate?

Linear Loop

\[
\begin{align*}
x &:= 2^{-10}, \quad y := 1 \\
\text{until } x &= y \text{ do} \\
\begin{bmatrix} x \\ y \end{bmatrix} &:= \begin{bmatrix} 2 & 0 \\ 7/4 & 1/4 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}
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\]

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\[
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until \( x \geq y \) do

\[
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y
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y
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Linear Loop

\[ x := 2^{-10}, \quad y := 1 \]

until \( x^2 y \geq 1 \) do

\[
\begin{bmatrix}
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  y
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- (semi-)algebraic sets
  \( \exists n \in \mathbb{N}. p(A^n x_0) \geq 0 \)
Termination Linear Loops

Does this loop terminate?

Linear Loop

\[
x := 2^{-10}, \ y := 1
\]

until \( x^2 y \geq 1 \) or \( x = y \) do

\[
\begin{bmatrix} x \\ y \end{bmatrix} := \begin{bmatrix} 2 & 0 \\ \frac{7}{4} & \frac{1}{4} \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}
\]
Termination Linear Loops

Does this loop terminate?

Linear Loop

\[
x \in [0, 1], \ y \in [1, 2]
\]

until \( \phi(x) \) do

\[
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- (semi-)algebraic sets
  \( \exists n \in \mathbb{N} p(A^n x_0) \geq 0 \)
- boolean combinations
- replace \( x_0 \) by an initial set \( \mathcal{X} \)
  \( \exists x_0 \in \mathcal{X} \exists n \in \mathbb{N} A^n x_0 \in S \)
  \( \forall x_0 \in \mathcal{X} \exists n \in \mathbb{N} A^n x_0 \in S \)
What is decidable about linear loops?

**Problem:** given $x_0$, $A$ and $S$, decide if $\exists n \in \mathbb{N}$ such that $A^nx_0 \in S$.

**Theorem (Orbit problem; Kannan and Lipton 1980, 1986)**
Decidable in polynomial time when $S$ is a singleton.

Already nontrivial proof using algebraic number theory!

**Theorem (Chonev, Ouaknine and Worrell, 2016)**
Decidable (in $\text{NP}^{\text{RP}}$) when $S$ is a linear subspace of dimension $\leq 3$.

Decidable (in $\text{PSPACE}$) when $S$ is a polytope of dimension $\leq 3$.

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Why do we need the dimension to be small?
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Problem: given $\mathcal{X}$, $A$ and $S$, decide if $\exists n \in \mathbb{N}$ such that $A^n \mathcal{X} \cap S \neq \emptyset$.

Theorem (Almagor, Ouaknine and Worrell, 2017)

Decidable (in $PSPACE$) when $\mathcal{X}, S$ are polytopes of dimension $\leq 3$. 
What is decidable about linear loops?

**Problem:** given $x_0$, $A$ and $S$, decide if $\exists n \in \mathbb{N}$ such that $A^n x_0 \in S$.

**Theorem (Orbit problem; Kannan and Lipton 1980, 1986)**

*Decidable in polynomial time when $S$ is a singleton.*

Already nontrivial proof using algebraic number theory!

**Theorem (Chonev, Ouaknine and Worrell, 2016)**

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*Decidable (in $PSPACE$) when $\mathcal{X}, S$ are polytopes of dimension $\leq 3$.*

Why do we need the dimension to be small?
From loops to recurrent sequences

### Linear Loop

\[
\begin{align*}
x & := x_0 \\
\text{until } 3x_1 - 7x_2 + 4x_3 & = 0 \text{ do} \\
x & := Ax
\end{align*}
\]
From loops to recurrent sequences

### Linear Loop

\[
x := x_0 \\
\text{until } y^T x = 0 \text{ do } x := Ax
\]

### Half-space reachability

Given \( x, y \in \mathbb{Q}^d \), \( A \in \mathbb{Q}^{d \times d} \), decide if \( \exists n \in \mathbb{N}. y^T A^n x_0 = 0 \).
From loops to recurrent sequences

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Consider the sequence \( u_n = y^T A^n x \).

Lemma

There exists \( a_0, \ldots, a_{d-1} \in \mathbb{Q} \) such that

\[ u_{n+d} = a_{d-1} u_{n+d-1} + \cdots + a_0 u_n, \quad \forall n \in \mathbb{N}. \]

In other words, \( (u_n)_n \) is a linear recurrent sequence (LRS).
From loops to recurrent sequences

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In other words, \( (u_n)_n \) is a linear recurrent sequence (LRS).

- Fibonacci: \( F_{n+2} = F_{n+1} + F_n \)
- Pell numbers: \( P_{n+2} = 2P_{n+1} + P_n \)
- very common in combinatorics
## From loops to recurrent sequences

### Linear Loop

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x & := x_0 \\
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In other words, \( (u_n)_n \) is a linear recurrent sequence (LRS). Conversely,

### Lemma

For any LRS \( (u_n)_n \), there exists \( x_0, y \) and \( A \) such that \( u_n = y^T A^n x_0 \).
Skolem and positivity problems

Linear recurrent sequence (LRS) of order $d$:

$$u_{n+d} = a_{d-1}u_{n+d-1} + \cdots + a_0u_n, \quad \forall n \in \mathbb{N}.$$ 

Remark: entirely determined by $u_0, \ldots, u_{d-1}$ and $a_0, \ldots, a_{d-1}$
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Given a LRS $(u_n)_n$, decide if $u_n = 0$ for some $n \in \mathbb{N}$.

This problem has been open for 70 years!
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Given a LRS \( (u_n)_n \), decide if \( u_n \geq 0 \) for all \( n \in \mathbb{N} \).

Harder than Skolem
Skolem Problem

Given a LRS \((u_n)_n\), decide if \(u_n = 0\) for some \(n \in \mathbb{N}\).

Theorem (Skolem, Mahler, and Lech, 1933, 1953, 1957)

The set \(\{n \in \mathbb{N} : u_n = 0\}\) is a union of finitely arithmetic progression and a finite set.
Skolem-Mahler-Lech theorem

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Theorem (Skolem, Mahler, and Lech, 1933, 1953, 1957)

*The set \(\{n \in \mathbb{N} : u_n = 0\}\) is a union of finitely arithmetic progression and a finite set.*

The regular pattern is computable. Nothing is known about the finite set: the proof is nonconstructive and uses \(p\)-adic analysis.
Skolem in low dimension

Theorem (Mignotte, Shorey, Tijdeman; Vereshchagin, 1985)

The Skolem problem is decidable for LRS of order 4.

Theorem (Blondel and Portier, 2002)

The Skolem problem is NP-hard.

For any \( x \in \mathbb{R} \), the (homogeneous Diophantine approximation) type

\[
L(x) = \inf_{n, c \in \mathbb{R}}: x - \frac{n}{m} < \frac{c}{m^2}
\]

for some \( n, m \in \mathbb{Z} \).

Intuitively, if \( L(x) > 0 \) then \( x \) is badly approximable by rationals.

Almost nothing known for any concrete \( x \) except that \( L(x) \in [0, 1/\sqrt{5}] \).

Theorem (Ouaknine and Worrell, 2013)

If Skolem is decidable at order 5 then one can approximate \( L(x) \) with arbitrary precision for a large class of numbers \( x \).
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How can we show hardness without proving undecidability?
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Given a LRS $(u_n)_n$, decide if $u_n \geq 0$ for all $n \in \mathbb{N}$.

### Theorem (Laohakosol and Tangsupphathawat, 2009)

The positivity problem is decidable at order $3$.

### Ultimate Positivity Problem

Given a LRS $(u_n)_n$, decide if $\exists N \in \mathbb{N}$ such that $u_n \geq 0$ for all $n \geq N$.

### Theorem (Ouaknine and Worrell, 2014)

The ultimate positivity problem is decidable for simple \textsuperscript{‡} LRS. It is at least as hard as deciding $\exists R$.
### Positivity Problem

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Positivity and eventual positivity

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The ultimate positivity problem is decidable for simple\(^\dagger\) LRS. It is at least as hard as deciding \(\exists R\).

\(^\dagger\)The associated characteristic polynomial has no repeated roots.
First-order queries on orbits

First-order orbit query (FOOQ): fully quantified first-order sentence whose atomic proposition are of the form

\[ p(x) \geq 0, \quad A^n x \in T \quad (T \text{ semialgebraic set}). \]
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Theorem (Almagor, Ouaknine and Worrell, 2021)

*Given A and \( \Phi(n) \) a FOOQ, it is decidable whether \( \exists n \in \mathbb{N}. \Phi(n) \) in dimension \( \leq 3 \).*
Given $x \in \mathbb{Q}^d$ and $A \in \mathbb{Q}^{n \times n}$ and $\mathcal{T}_1, \ldots, \mathcal{T}_k \subseteq \mathbb{R}^d$ semialgebraic sets.
Given \( x \in \mathbb{Q}^d \) and \( A \in \mathbb{Q}^{n \times n} \) and \( T_1, \ldots, T_k \subseteq \mathbb{R}^d \) semialgebraic sets. Let \( \Sigma = \{0, 1\}^k \) and define \( w \in \Sigma^\mathbb{N} \) by

\[
w_n = (A^n x \in T_1, \ldots, A^n x \in T_k).
\]

Intuition: \( w_n \) records to which sets \( A^n x \) belongs to at each step \( n \).
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Problem: given an MSO formula \( \Psi \) over \((\mathbb{N}, <)\), decide whether \( w \models \Psi \).

Examples: \( P_i(n) \) means \( A^n x \in T_i \)
- \( T_i \) is reachable: \( \exists n. P_i(n) \)
- whenever \( T_i \) is visited \( T_j \) is visited some point later:

\[
\forall n : P_i(n) \Rightarrow (\exists m > n : P_j(m))
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MSO model-checking

Given $x \in \mathbb{Q}^d$ and $A \in \mathbb{Q}^{n \times n}$ and $T_1, \ldots, T_k \subseteq \mathbb{R}^d$ semialgebraic sets. Let $\Sigma = \{0, 1\}^k$ and define $w \in \Sigma^\mathbb{N}$ by

$$w_n = (A^n x \in T_1, \ldots, A^n x \in T_k).$$

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- in target $T_i$ at every odd position:

$$\exists O \subseteq \mathbb{N} : \text{formula to define odd numbers} \land \forall x : x \in O \Rightarrow P_i(x)$$
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Theorem (Karimov, Lefaucheux, Ouaknine, Purser, Varonka, Whiteland, Worrell)

This is decidable if all \( T_i \) either have intrinsic dimension 1 or are included in a subspace of dimension 3.

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Continuous linear dynamical systems

Linear differential equation:

\[ x'(t) = Ax(t) \]

\[ x(0) = x_0 \]
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Example:

\[ x'(t) = 7x(t) \]

\[ \Rightarrow x(t) = e^{7t} \]
Continuous linear dynamical systems

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Example:

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\[ \begin{cases} x'_1(t) = x_2(t) \\ x'_2(t) = -x_1(t) \end{cases} \iff \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}' = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \]

\[ \leadsto x(t) = e^{7t} \]

\[ \leadsto \begin{cases} x_1(t) = \sin(t) \\ x_2(t) = \cos(t) \end{cases} \]
Continuous linear dynamical systems

Linear differential equation:

\[ x'(t) = Ax(t) \quad x(0) = x_0 \]

General solution form:

\[ x(t) = e^{At} x_0 \]

where \( e^M = \sum_{n=0}^{\infty} \frac{M^n}{n!} \)
Continuous reachability

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Given $x, y$ and $A$, decide if $\exists t \in \mathbb{R}$ such that $x^T e^{At} y = 0$.

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Given $x, y$ and $A$, decide whether $x^T e^{At} y \geq 0$ for all $t \geq 0$.

Continuous positivity is inter-reducible with continuous Skolem.
The decidability of all these problems is also open!
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Given $x$, $y$ and $A$, decide if $\exists t \in [0, 1]$ such that $x^T e^{At} y = 0$.

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A link with number theory

Some reachability questions look like this:

\[ \exists t \in \mathbb{R}. \ 42t^7 = 56 \land e^{3t} - e^t = 9 \]
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Transcendental number: not algebraic, e.g. \( e, \pi \)
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Theorem (Special case of Lindemann–Weierstrass)

*If $t$ is a nonzero algebraic number then $e^t$ is transcendental.*
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Theorem (Special case of Lindemann–Weierstrass)

*If* \( t \) *is a nonzero algebraic number then* \( e^t \) *is transcendental.*

- \( P\left(t\right) = 0 \) so \( t \) is algebraic (by definition)
- Lindemann–Weierstrass: \( e^t \) transcendental (unless \( t = 0 \))
- hence \( Q\left(e^t\right) \neq 0 \) (except maybe if \( t = 0 \))
Exponential polynomial

In general,

\[ x^T e^{A t} y = \sum_{i=1}^{d} P_i(t) e^{\lambda_i t} \]

where \( P_i \) polynomial, \( \lambda_i \in \mathbb{C} \) eigenvalues of \( A \).
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Theorem (Wilkie and MacIntyre)

If Schanuel’s conjecture is true, then, for each \( k \in \mathbb{N} \), the first-order theory of the structure \( (\mathbb{R}, 0, 1, <, +, \cdot, \exp, \cos \upharpoonright_{[0,k]}, \sin \upharpoonright_{[0,k]}) \) is decidable.

▶ algorithm always correct, only termination requires the conjecture
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- algorithm always correct, only termination requires the conjecture
- this makes many problem (inc. continuous Skolem) decidable!
Exponential polynomial

In general,

\[ x^T e^{At} y = \sum_{i=1}^{d} P_i(t) e^{\lambda_i t} \]

where \( P_i \) polynomial, \( \lambda_i \in \mathbb{C} \) eigenvalues of \( A \).

Lindemann–Weierstrass’s theorem is not enough to solve the continuous Skolem problem.

**Theorem (Wilkie and MacIntyre)**

*If Schanuel’s conjecture is true, then, for each \( k \in \mathbb{N} \), the first-order theory of the structure \((\mathbb{R}, 0, 1, <, +, \cdot, \exp, \cos \upharpoonright [0,k], \sin \upharpoonright [0,k])\) is decidable.*

▶ algorithm always correct, only termination requires the conjecture
▶ this makes many problem (inc. continuous Skolem) decidable!

What is Schanuel’s conjecture?
If \( z_1, \ldots, z_n \) that are \textbf{linearly independent} over \( \mathbb{Q} \), then at least \( n \) numbers among \( z_1, \ldots, z_n, e^{z_1}, \ldots, e^{z_n} \) are \textbf{algebraically independent}.
Schanuel’s conjecture

If \( z_1, \ldots, z_n \) that are \textbf{linearly independent} over \( \mathbb{Q} \), then at least \( n \) numbers among \( z_1, \ldots, z_n, e^{z_1}, \ldots, e^{z_n} \) are \textbf{algebraically independent}.

**Example:** \( \pi \) and \( e \) are algebraically independent

\[
\begin{align*}
z_1 &= i\pi, \quad z_2 = 1 \\
\therefore \quad e^{z_1} &= -1, \quad e^{z_2} = e.
\end{align*}
\]
Schanuel’s conjecture

If $z_1, \ldots, z_n$ that are linearly independent over $\mathbb{Q}$, then at least $n$ numbers among $z_1, \ldots, z_n, e^{z_1}, \ldots, e^{z_n}$ are algebraically independent.

Example: $\pi$ and $e$ are algebraically independent

$$z_1 = i\pi, \quad z_2 = 1 \quad \leadsto \quad e^{z_1} = -1, \quad e^{z_2} = e.$$ 

Clearly $z_1$ and $z_2$ are linearly independent over $\mathbb{Q}$. So at least 2 of $i\pi, 1, -1, e$ are algebraically independent. But 1 is algebraic so $\pi$ and $e$ are algebraically independent.
Schanuel’s conjecture

If $z_1, \ldots, z_n$ that are linearly independent over $\mathbb{Q}$, then at least $n$ numbers among $z_1, \ldots, z_n, e^{z_1}, \ldots, e^{z_n}$ are algebraically independent.

Example: $\pi$ and $e$ are algebraically independent

$$z_1 = i\pi, \ z_2 = 1 \quad \leadsto \quad e^{z_1} = -1, \ e^{z_2} = e.$$  

Clearly $z_1$ and $z_2$ are linearly independent over $\mathbb{Q}$. So at least 2 of $i\pi, 1, -1, e$ are algebraically independent. But 1 is algebraic so $\pi$ and $e$ are algebraically independent.

Summary:

- Schanuel implies that $\pi, e, \pi + e, e\pi, \ldots$ are transcendental.
- $\pi$ and $e$ are known to be transcendental
- $\pi + e$ is not known to be transcendental
Bounded continuous Skolem problem: given $x, y$ and $A$, decide if
- unbounded: $\exists t \in [0, 1]$ such that $x^T e^{At} y = 0$.
- bounded: $\exists t \in \mathbb{R}$ such that $x^T e^{At} y = 0$.

Theorem (Chonev, Ouaknine and Worrell, 2016)

*The bounded continuous Skolem Problem is decidable subject to Schanuel's conjecture.*
Continuous reachability

Bounded continuous Skolem problem: given $x$, $y$ and $A$, decide if

- **unbounded:** $\exists t \in [0, 1]$ such that $x^T e^{At} y = 0$.
- **bounded:** $\exists t \in \mathbb{R}$ such that $x^T e^{At} y = 0$.

Theorem (Chonev, Ouaknine and Worrell, 2016)

The bounded continuous Skolem Problem is decidable **subject to Schanuel’s conjecture**.

Theorem (Chonev, Ouaknine and Worrell, 2016)

If the (unbounded) continuous Skolem Problem is decidable then the Diophantine-approximation types of all real algebraic numbers is computable.

In other words: it requires new mathematics...
More complicated programs

Linear loop with if

\[
x := 2^{-10}
\]
\[
y := 1
\]
while \( y \geq x \) do
  if \( y \geq 2x \) then
    \[
    \begin{bmatrix}
    x \\
    y
    \end{bmatrix}
    :=
    \begin{bmatrix}
    2 & 0 \\
    1 & 4
    \end{bmatrix}
    \begin{bmatrix}
    x \\
    y
    \end{bmatrix}
    
  \]
  else
    \[
    \begin{bmatrix}
    x \\
    y
    \end{bmatrix}
    :=
    \begin{bmatrix}
    2 & 3 \\
    -3 & 7
    \end{bmatrix}
    \begin{bmatrix}
    x \\
    y
    \end{bmatrix}
    \]
  \]
Linear loop with if

\[
\begin{align*}
x &:= 2^{-10} \\
y &:= 1 \\
\text{while } y \geq x \text{ do} \\
& \quad \text{if } y \geq 2x \text{ then} \\
& \quad \quad \begin{bmatrix} x \\ y \end{bmatrix} := \begin{bmatrix} 2 & 0 \\ 1 & 4 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} \\
& \quad \text{else} \\
& \quad \quad \begin{bmatrix} x \\ y \end{bmatrix} := \begin{bmatrix} 2 & 3 \\ -3 & 7 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}
\end{align*}
\]

Reachability is trivially undecidable by simulating two counter automata
More complicated programs

**Linear loop with if**

\[ x := 2^{-10} \]
\[ y := 1 \]

while \( y \geq x \) do

if \( y \geq 2x \) then

\[
\begin{bmatrix}
  x \\
  y
\end{bmatrix} := \begin{bmatrix} 2 & 0 \\ 1 & 4 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}
\]

else

\[
\begin{bmatrix}
  x \\
  y
\end{bmatrix} := \begin{bmatrix} 2 & 3 \\ -3 & 7 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}
\]

**Nondeterminic loop**

\[ x := 2^{-10} \]
\[ y := 1 \]

while true do

non deterministically do

\[
\begin{bmatrix}
  x \\
  y
\end{bmatrix} := \begin{bmatrix} 2 & 0 \\ 1 & 4 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}
\]

or

\[
\begin{bmatrix}
  x \\
  y
\end{bmatrix} := \begin{bmatrix} 2 & 3 \\ -3 & 7 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}
\]

Reachability is trivially undecidable by simulating two counter automata
More complicated programs

Linear loop with if

\[
\begin{align*}
x &:= 2^{-10} \\
y &:= 1 \\
\text{while } y \geq x \text{ do} \\
&\quad \text{if } y \geq 2x \text{ then} \\
&\quad \quad \begin{bmatrix} x \\ y \end{bmatrix} := \begin{bmatrix} 2 & 0 \\ 1 & 4 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} \\
&\quad \text{else} \\
&\quad \quad \begin{bmatrix} x \\ y \end{bmatrix} := \begin{bmatrix} 2 & 3 \\ -3 & 7 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}
\end{align*}
\]

Nondeterminic loop

\[
\begin{align*}
x &:= 2^{-10} \\
y &:= 1 \\
\text{while } \text{true} \text{ do} \\
&\quad \text{non deterministically do} \\
&\quad \quad \begin{bmatrix} x \\ y \end{bmatrix} := \begin{bmatrix} 2 & 0 \\ 1 & 4 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} \\
&\quad \quad \text{or} \\
&\quad \quad \begin{bmatrix} x \\ y \end{bmatrix} := \begin{bmatrix} 2 & 3 \\ -3 & 7 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}
\end{align*}
\]

Reachability is trivially undecidable by simulating two counter automata

- Overapproximate behaviours
- Nondeterminic
Example: 2D robot

State: \( \mathbf{\tilde{u}} = (x_\theta, y_\theta, x, y) \)

Discretized actions:
- rotate arm by \( \psi \)
- change arm length by \( \delta \)

\[ \begin{align*}
\text{Linear transformations} \\
\text{Rotate arm by } & \psi: \\
\begin{pmatrix} x \\ y \end{pmatrix} & \leftarrow \begin{pmatrix} \cos \psi & -\sin \psi \\ \sin \psi & \cos \psi \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \\
\begin{pmatrix} x_\theta \\ y_\theta \end{pmatrix} & \leftarrow \begin{pmatrix} \cos \psi & -\sin \psi \\ \sin \psi & \cos \psi \end{pmatrix} \begin{pmatrix} x_\theta \\ y_\theta \end{pmatrix} \\
\text{Change arm length by } & \delta: \\
\begin{pmatrix} x \\ y \end{pmatrix} & \leftarrow \begin{pmatrix} x \\ y \end{pmatrix} + \delta \begin{pmatrix} x_\theta \\ y_\theta \end{pmatrix}
\end{align*} \]
Matrix problems

Input: $A, C \in \mathbb{Q}^{d \times d}$ matrices
Output: $\exists n \in \mathbb{N}$ such that $A^n = C$ ?

Example: $\exists n \in \mathbb{N}$ such that

\[
\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}^n = \begin{bmatrix} 1 & 100 \\ 0 & 1 \end{bmatrix} ?
\]
Matrix problems

Input: \( A, C \in \mathbb{Q}^{d\times d} \) matrices  
Output: \( \exists n \in \mathbb{N} \) such that \( A^n = C \)  
✓ Decidable (PTIME)

Example: \( \exists n \in \mathbb{N} \) such that  
\[
\begin{bmatrix}
1 & 1 \\
0 & 1
\end{bmatrix}^n = \begin{bmatrix} 1 & 100 \\
0 & 1 \end{bmatrix} \]
Matrix problems

Input: $A, C \in \mathbb{Q}^{d \times d}$ matrices
Output: $\exists n \in \mathbb{N}$ such that $A^n = C$ ? ✓ Decidable (PTIME)

Input: $A, B, C \in \mathbb{Q}^{d \times d}$ matrices
Output: $\exists n, m \in \mathbb{N}$ such that $A^n B^m = C$ ?

Example: $\exists n, m \in \mathbb{N}$ such that

\[
\begin{bmatrix} 2 & 3 \\ 0 & 1 \end{bmatrix}^n \begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \end{bmatrix}^m = \begin{bmatrix} 1 & 60 \\ 0 & 1 \end{bmatrix}
\]
Matrix problems

Input: \( A, C \in \mathbb{Q}^{d \times d} \) matrices
Output: \( \exists n \in \mathbb{N} \) such that \( A^n = C \) ? ✓ Decidable (PTIME)

Input: \( A, B, C \in \mathbb{Q}^{d \times d} \) matrices
Output: \( \exists n, m \in \mathbb{N} \) such that \( A^n B^m = C \) ? ✓ Decidable

Example: \( \exists n, m \in \mathbb{N} \) such that
\[
\begin{bmatrix} 2 & 3 \\ 0 & 1 \end{bmatrix}^n \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ 0 & 1 \end{bmatrix}^m = \begin{bmatrix} 1 & 60 \\ 0 & 1 \end{bmatrix}
\] ?
Matrix problems

Input: $A, C \in \mathbb{Q}^{d \times d}$ matrices
Output: $\exists n \in \mathbb{N}$ such that $A^n = C$ ?  ✓ Decidable (PTIME)

Input: $A, B, C \in \mathbb{Q}^{d \times d}$ matrices
Output: $\exists n, m \in \mathbb{N}$ such that $A^n B^m = C$ ?  ✓ Decidable

Input: $A_1, \ldots, A_k, C \in \mathbb{Q}^{d \times d}$ matrices
Output: $\exists n_1, \ldots, n_k \in \mathbb{N}$ such that $\prod_{i=1}^k A_i^{n_i} = C$ ?

Example: $\exists n, m, p \in \mathbb{N}$ such that

\[
\begin{bmatrix}
2 & 3 \\
0 & 1 \\
\end{bmatrix}^n \begin{bmatrix}
\frac{1}{2} & \frac{1}{2} \\
0 & 1 \\
\end{bmatrix}^m \begin{bmatrix}
2 & 5 \\
0 & 1 \\
\end{bmatrix}^p = \begin{bmatrix}
81 & 260 \\
0 & 1 \\
\end{bmatrix} ?
\]
Matrix problems

Input: $A, C \in \mathbb{Q}^{d \times d}$ matrices
Output: $\exists n \in \mathbb{N}$ such that $A^n = C$ ?
✓ Decidable (PTIME)

Input: $A, B, C \in \mathbb{Q}^{d \times d}$ matrices
Output: $\exists n, m \in \mathbb{N}$ such that $A^n B^m = C$ ?
✓ Decidable

Input: $A_1, \ldots, A_k, C \in \mathbb{Q}^{d \times d}$ matrices
Output: $\exists n_1, \ldots, n_k \in \mathbb{N}$ such that $\prod_{i=1}^k A_i^{n_i} = C$ ?
✓ Decidable if $A_i$ commute × Undecidable in general

Example: $\exists n, m, p \in \mathbb{N}$ such that

\[
\begin{bmatrix}
2 & 3 \\
0 & 1 \\
\end{bmatrix}^n \quad \begin{bmatrix}
\frac{1}{2} & \frac{1}{2} \\
0 & 1 \\
\end{bmatrix}^m \quad \begin{bmatrix}
2 & 5 \\
0 & 1 \\
\end{bmatrix}^p = \begin{bmatrix}
81 & 260 \\
0 & 1 \\
\end{bmatrix}
? \]
Matrix problems

Input: $A, C \in \mathbb{Q}^{d \times d}$ matrices
Output: $\exists n \in \mathbb{N}$ such that $A^n = C$ ?  ✓ Decidable (PTIME)

Input: $A, B, C \in \mathbb{Q}^{d \times d}$ matrices
Output: $\exists n, m \in \mathbb{N}$ such that $A^n B^m = C$ ? ✓ Decidable

Input: $A_1, \ldots, A_k, C \in \mathbb{Q}^{d \times d}$ matrices
Output: $\exists n_1, \ldots, n_k \in \mathbb{N}$ such that $\prod_{i=1}^k A_i^{n_i} = C$ ? ✓ Decidable if $A_i$ commute × Undecidable in general

Input: $A_1, \ldots, A_k, C \in \mathbb{Q}^{d \times d}$ matrices
Output: $C \in \langle \text{semigroup generated by } A_1, \ldots, A_k \rangle$ ?

Semigroup: $\langle A_1, \ldots, A_k \rangle = \text{all finite products of } A_1, \ldots, A_k$
Examples:

$$A_1 A_3 A_2 \quad A_1 A_2 A_1 A_2 \quad A_3^8 A_2 A_1^3 A_3^{42}$$
Matrix problems

Input: \( A, C \in \mathbb{Q}^{d \times d} \) matrices
Output: \( \exists n \in \mathbb{N} \) such that \( A^n = C \) ?
✓ Decidable (PTIME)

Input: \( A, B, C \in \mathbb{Q}^{d \times d} \) matrices
Output: \( \exists n, m \in \mathbb{N} \) such that \( A^n B^m = C \) ?
✓ Decidable

Input: \( A_1, \ldots, A_k, C \in \mathbb{Q}^{d \times d} \) matrices
Output: \( \exists n_1, \ldots, n_k \in \mathbb{N} \) such that \( \prod_{i=1}^{k} A_i^{n_i} = C \) ?
✓ Decidable if \( A_i \) commute ∙ Undecidable in general

Input: \( A_1, \ldots, A_k, C \in \mathbb{Q}^{d \times d} \) matrices
Output: \( C \in \langle \text{semigroup generated by } A_1, \ldots, A_k \rangle \) ?
✓ Decidable if \( A_i \) commute ∙ Undecidable in general

Semigroup: \( \langle A_1, \ldots, A_k \rangle = \) all finite products of \( A_1, \ldots, A_k \)
Examples:
\[
A_1 A_3 A_2 \quad A_1 A_2 A_1 A_2 \quad A_3^8 A_2 A_1^3 A_3^{42}
\]
Discrete reachability problems

Every nontrivial extension of simple linear loops seems to lead to undecidable problems.
Every nontrivial extension of simple linear loops seems to lead to undecidable problems. *What about the continuous setting?*
RC circuit

\[ I_{\text{CLOSED}} = 1/RC (V - V_C) \]

\[ I_R = -1/RC \]

\[ V_R = -1/C \]

\[ I = 0 \]

\[ I_R = -1/R V \]

\[ V_C = 1/C \]

\[ t \]
OPEN

\[
\begin{align*}
\dot{i} &= 0 \\
\dot{i}_R &= -\frac{1}{RC} i_R \\
\dot{V}_R &= -\frac{1}{C} i_R \\
\dot{Q} &= i_R \\
\dot{V}_C &= \frac{1}{C} i_R
\end{align*}
\]
RC circuit

\[ \dot{I} = 0 \]
\[ \dot{I}_R = -\frac{1}{RC} I_R \]
\[ \dot{V}_R = -\frac{1}{C} I_R \]
\[ Q = I_R \]
\[ \dot{V}_C = \frac{1}{C} I_R \]

\[ \dot{I} = -\frac{1}{RC} I_R \]
\[ \dot{I}_R = -\frac{1}{RC} I_R \]
\[ \dot{V}_R = -\frac{1}{C} I_R \]
\[ Q = I_R \]
\[ \dot{V}_C = \frac{1}{C} I_R \]
RC circuit

\[ I \quad \text{CLOSED} \quad I_R \quad V_R \quad \text{OPEN} \quad V \quad Q \quad C \quad V_C \]

**OPEN**

\[
\begin{align*}
\dot{I} &= 0 \\
\dot{I_R} &= -\frac{1}{RC} I_R \\
\dot{V_R} &= -\frac{1}{C} I_R \\
Q &= I_R \\
\dot{V_C} &= \frac{1}{C} I_R
\end{align*}
\]

**CLOSED**

\[
\begin{align*}
I &= \frac{1}{R} (V - V_C) \\
I_R &= \frac{1}{R} (V - V_C) \\
V_R &= V - V_C \\
\dot{I} &= 0 \\
\dot{I_R} &= -\frac{1}{RC} I_R \\
\dot{V_R} &= -\frac{1}{C} I_R \\
Q &= I_R \\
\dot{V_C} &= \frac{1}{C} I_R
\end{align*}
\]
Switching systems

Restricted hybrid system:
- linear dynamics
- no guards (nondeterministic)
- no discrete updates

\[
\begin{align*}
    x' &= A_1 x \\
    x' &= A_2 x \\
    x' &= A_3 x \\
    x' &= A_4 x
\end{align*}
\]
Switching systems

Restricted hybrid system:
- linear dynamics
- no guards (nondeterministic)
- no discrete updates

Dynamics:
\[ e^{A_4 t_4} e^{A_3 t_3} e^{A_2 t_2} e^{A_1 t_1} \]
Switching systems

Restricted hybrid system:
- linear dynamics
- no guards (nondeterministic)
- no discrete updates

Problem:

\[ e^{A_4 t_4} e^{A_3 t_3} e^{A_2 t_2} e^{A_1 t_1} = C ? \]

What we control: \( t_1, t_2, t_3, t_4 \in \mathbb{R}_{\geq 0} \)
Related work in the continuous case

Input: $A, C \in \mathbb{Q}^{d \times d}$ matrices
Output: $\exists t \in \mathbb{R}$ such that $e^{At} = C$ ?

Example: $\exists t \in \mathbb{R}$ such that

$$
\exp \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} t = \begin{pmatrix} 1 & 100 \\ 0 & 1 \end{pmatrix}
$$
Related work in the continuous case

Input: $A, C \in \mathbb{Q}^{d \times d}$ matrices
Output: $\exists t \in \mathbb{R}$ such that $e^{At} = C$ ? ✓ Decidable (PTIME)

Example: $\exists t \in \mathbb{R}$ such that

$$\exp \left( \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} t \right) = \begin{bmatrix} 1 & 100 \\ 0 & 1 \end{bmatrix} ?$$
Related work in the continuous case

Input: $A, C \in \mathbb{Q}^{d \times d}$ matrices
Output: $\exists t \in \mathbb{R}$ such that $e^{At} = C$  ✓ Decidable (PTIME)

Input: $A, B, C \in \mathbb{Q}^{d \times d}$ matrices
Output: $\exists t, u \in \mathbb{N}$ such that $e^{At} e^{Bu} = C$

Example: $\exists t, u \in \mathbb{R}$ such that

$$\exp \left( \begin{bmatrix} 2 & 3 \\ 0 & 1 \end{bmatrix} t \right) \exp \left( \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ 0 & 1 \end{bmatrix} u \right) = \begin{bmatrix} 1 & 60 \\ 0 & 1 \end{bmatrix}$$
Related work in the continuous case

Input: $A, C \in \mathbb{Q}^{d \times d}$ matrices  
Output: $\exists t \in \mathbb{R}$ such that $e^{At} = C$  
✓ Decidable (PTIME)

Input: $A, B, C \in \mathbb{Q}^{d \times d}$ matrices  
Output: $\exists t, u \in \mathbb{N}$ such that $e^{At}e^{Bu} = C$  
× Unknown

Example: $\exists t, u \in \mathbb{R}$ such that

$$
\exp \begin{pmatrix} 2 & 3 \\ 0 & 1 \end{pmatrix} t \exp \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ 0 & 1 \end{pmatrix} u = \begin{pmatrix} 1 & 60 \\ 0 & 1 \end{pmatrix}
$$

?
Switching system

What about a loop?
Switching system

\[ x' = A_1 x \rightarrow x' = A_2 x \rightarrow x' = A_4 x \rightarrow x' = A_3 x \]

What about a loop?

Dynamics:

\[ e^{A_4 t'_4} e^{A_3 t'_3} e^{A_2 t'_2} e^{A_1 t'_1} e^{A_4 t_4} e^{A_3 t_3} e^{A_2 t_2} e^{A_1 t_1} \]
Switching system

\[ x' = A_1 x \quad \text{Loop} \iff \text{clique} \quad x' = A_4 x \]

\[ x' = A_2 x \quad x' = A_3 x \]

Remark:

zero time dynamics \((t_i = 0)\) are allowed
Switching system

\[ x' = A_1 x \quad \rightarrow \quad x' = A_2 x \]

\[ x' = A_4 x \quad \leftarrow \quad x' = A_3 x \]

Dynamics:

any finite product of \( e^{A_i t} \) \( \sim \) semigroup!
Switching system

Problem:

\[ C \in \mathcal{G} \quad ? \]

where

\[ \mathcal{G} = \langle \text{semigroup generated by } e^{A_i t} \text{ for all } t \geq 0 \rangle \]
Reachability for switching systems

Input: $A_1, \ldots, A_k, C \in \mathbb{Q}^{d \times d}$ matrices
Output: $\exists t_1, \ldots, t_k \geq 0$ such that

$$\prod_{i=1}^{n} e^{A_it_i} = C \ ?$$

Input: $A_1, \ldots, A_k, C \in \mathbb{Q}^{d \times d}$ matrices
Output:

$$C \in \langle \text{semigroup generated by } e^{A_1t}, \ldots, e^{A_kt} : t \geq 0 \rangle \ ?$$

Theorem (Ouaknine, P, Sous-Pinto, Worrell)

Both problems are:

- **Undecidable** in general
- **Decidable** when all the $A_i$ commute
Some words about the proof (commuting case)

<table>
<thead>
<tr>
<th>Product Problem</th>
<th>Semigroup Problem</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \exists t_1, \ldots, t_k \geq 0 ) s.t. ( \prod_{i=1}^{n} e^{A_i t_i} = C )</td>
<td>( C \in \langle e^{A_1 t}, \ldots, e^{A_k t} : t \geq 0 \rangle )</td>
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Reduce

<table>
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<tr>
<td>( \exists n \in \mathbb{Z}^d ) s.t. ( \pi Bn \leq s )</td>
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Some words about the proof (commuting case)

Product Problem

\[ \exists t_1, \ldots, t_k \geq 0 \text{ s.t. } \prod_{i=1}^{n} e^{A_i t_i} = C \]

Semigroup Problem

\[ C \in \langle e^{A_1 t}, \ldots, e^{A_k t} : t \geq 0 \rangle \]

Integer Linear Programming

\[ \exists n \in \mathbb{Z}^d \text{ s.t. } \pi B n \leq s \]

\[ \exists n \in \mathbb{Z}^d \text{ s.t. } \pi B n \leq s \]

\[ s \text{ of the form: } a_0 + \log(a_1) + \cdots + \log(a_k) \]

\[ B, a_0, \ldots, a_k \text{ are algebraic} \]
Some words about the proof (commuting case)

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Introduce the commuting case

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Reduce to integer linear programming

How did we get from reals to integers with $\pi$?

$e^{it} = \alpha \iff t \in \log(\alpha) + 2\pi \mathbb{Z}$
Integer Linear Programming

\[ \exists n \in \mathbb{Z}^d \text{ such that } \pi Bn \leq s \ ? \]

where \( s \) is a linear form in logarithms of algebraic numbers
$\exists n \in \mathbb{Z}^d \text{ such that } \pi B n \leq s \ ?$

where $s$ is a linear form in logarithms of algebraic numbers

Key ingredient: Diophantine approximations

- Finding integer points in cones: Kronecker’s theorem
Integer Linear Programming

\[\exists n \in \mathbb{Z}^d \text{ such that } \pi Bn \leq s \quad ?\]

where \(s\) is a linear form in logarithms of algebraic numbers

Key ingredient: Diophantine approximations

- Finding integer points in cones: Kronecker’s theorem

\[\sqrt{2} + \log \sqrt{3} - 3 \log \sqrt{7} \quad ? \quad 1 + \log 9 - \log \sqrt[42]{666}\]
Some words about the proof (general case)

<table>
<thead>
<tr>
<th>Product Problem</th>
<th>Semigroup Problem</th>
</tr>
</thead>
<tbody>
<tr>
<td>[ \exists t_1, \ldots, t_k \geq 0 \text{ s.t. } \prod_{i=1}^{n} e^{A_i t_i} = C ]</td>
<td>[ C \in \langle e^{A_1 t}, \ldots, e^{A_k t} : t \geq 0 \rangle ]</td>
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<th>Hilbert’s Tenth Problem</th>
<th>Theorem (Matiyasevich)</th>
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<tr>
<td>[ \exists n \in \mathbb{Z}^d \text{ s.t. } p(n) = 0 ]</td>
<td><strong>Hilbert’s Tenth Problem is undecidable</strong></td>
</tr>
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</table>
Summary on reachability

Exact reachability is hard:

▶ Skolem/Positivity problem for linear loops (Open for 70 years)
▶ Every mild extension is undecidable
▶ Decidability requires very strong assumptions (commuting matrices)

Continuous vs discrete setting

▶ similar results
▶ different techniques
▶ continuous setting can leverage powerful results/conjectures
Control Theory
Example: mass-spring-damper system

State: \( X = z \in \mathbb{R} \)

Equation of motion:
\[
mz'' = -kz - bz' + mg + u
\]
Example: mass-spring-damper system

Model with external input \( u(t) \)

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\( \rightarrow \) Affine but not first order
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Equation of motion:

$$mz'' = -kz - bz' + mg + u$$

→ Affine but not first order

State: $X = (z, z', 1) \in \mathbb{R}^3$

Equation of motion:

$$
\begin{bmatrix}
z' \\
z'' \\
1
\end{bmatrix}
= 
\begin{bmatrix}
-kz - \frac{bz'}{m} + g + \frac{1}{m}u \\
-z' \\
0
\end{bmatrix}
$$
Example: mass-spring-damper system

Model with external input $u(t)$ → Linear time invariant system

$$X' = AX + Bu$$

with some constraints on $u$.

State: $X = z \in \mathbb{R}$

Equation of motion:

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$$
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0
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$$
A very simple example

A simplified one-dimensional car: control acceleration $u(t)$

$$x''(t) = u(t)$$
A very simple example

A simplified one-dimensional car: control acceleration $u(t)$

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Starting at $x(0) = a$, want to reach and stop at $x = b$: 

![Diagram of a car moving from a to b with velocity $x' = 0$ at both points](image-url)
A very simple example

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\[ x''(t) = u(t) \]

Starting at $x(0) = a$, want to reach and stop at $x = b$:

Possible solution:
A very simple example

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More realistic solution:
A very simple example

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Rephrasing the problem:

$$\begin{cases} x' = y \\ y' = u \end{cases} \iff \begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} 0 \\ u \end{bmatrix} \iff X' = AX + U$$

Starting from $(x, y) = (a, 0)$, try to reach $(x, y) = (b, 0)$.

This is a point-to-point reachability problem.
The problem

LTI Reachability problem

- a source \( y \in \mathbb{Q}^n \),
- a target \( z \in \mathbb{Q}^n \),
- a transition matrix \( A \in \mathbb{Q}^{n \times n} \),
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decide if \( \exists T \geq 0, \ u : [0, T] \rightarrow U \) measurable such that \( x(T) = z \) where

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**Warning:** \( u \) does not need to be “describable”, e.g. piecewise polynomial. Otherwise, completely changes the nature of the problem.
## Continuous Reachability problem

- a source $y \in \mathbb{Q}^n$
- a target $z \in \mathbb{Q}^n$
- a transition function $f$
- a set of controls $U \subseteq \mathbb{R}^m$

Decide if $\exists T \geq 0, \ u : [0, T] \to U$ measurable such that $x(T) = z$ where

$$x(0) = y, \quad x'(t) = f(t, x(t), u(t)) \quad \text{for} \ t \in [0, T].$$

Generally undecidable:

- for nonlinear systems, even without control ($U = \{0\}$)
- piecewise constant derivative systems (PCD), still no control
- linear saturated systems (at least for discrete systems), no control

LTI systems probably form the most useful class that is not undecidable. But do they really?
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- can all points \( y \in \mathbb{R}^n \) reach \( z = 0 \)? \hspace{1cm} \text{global null-controllability}
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But also:

- assumptions on \( A \) (typically spectral)
- assumptions on \( U \)
- restrictions on acceptable \( u \)
Two known extreme cases

▶ When we have no control:

\[ U = \{0\} \quad \text{and} \quad x'(t) = Ax + u(t) \quad \iff \quad x(t) = e^{At}x(0). \]
Two known extreme cases

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Theorem (Hainry’08)

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► When we can control in a vector space:

\[ U = B\mathbb{R}^m \quad \text{and} \quad x'(t) = Ax + u(t) \quad \Rightarrow \quad x(t) \in \text{span}[B, AB, \ldots, A^{n-1}B] \]
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Theorem (Folklore)

Given \( y, z \in \mathbb{Q}^n \) and \( A \in \mathbb{Q}^{n \times n} \), \( B \in \mathbb{Q}^{n \times m} \), it is decidable whether 
\( \exists T \geq 0 \) and \( u : [0, T] \to \mathbb{R}^m \) measurable such that \( x(0) = y \) and 
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A simplified one-dimensional car: control acceleration $u(t)$

$$x''(t) = u(t)$$

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Reality: acceleration/braking is not infinite $\leadsto u$ is bounded!
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Very few decidability results in the literature in this case.
Our results: decidability

LTI Zonotope Null-Reachability problem

Given a matrix $A \in \mathbb{Q}^{n \times n}$, a set of controls $U = B[-1, 1]^m$, a target $z \in \mathbb{Q}^n$, decide if $\exists T \geq 0, u : [0, T] \rightarrow U$ such that $x(T) = z$ where

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Theorem (Dantam, P.)

The LTI Zonotope Null-Reachability problem is decidable if one of:

- $A$ is real diagonal, $B$ is a column with at most 2 nonzero entries,
- $A$ is real diagonalizable, eigenvalues $\subseteq \alpha \mathbb{Q}$ for some $\alpha \in \mathbb{Q}$,
- $A$ only has one eigenvalue which is real, $B$ is a column,
- dimension $n = 2$, $B$ is a column and $A$ has real eigenvalues.

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### Our results: conditional decidability

#### Schanuel’s conjecture

A deep conjecture in **transcendental number theory**. Widely believed to be true and totally open.

#### Theorem (Dantam, P.)

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- A has **real eigenvalues**,
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*and Schanuel’s conjecture is true.*
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Theorem (Wilkie and MacIntyre)
If Schanuel’s conjecture is true, then, for each $k \in \mathbb{N}$, the first-order theory of the structure $(\mathbb{R}, 0, 1, <, +, \cdot, \exp, \cos  \upharpoonright [0,k], \sin  \upharpoonright [0,k])$ is decidable.
Hardness

Study generalization:
Hardness

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LTI Null-Set-Reachability problem

Given a matrix $A \in \mathbb{Q}^{n \times n}$, a set of controls $U \subseteq \mathbb{R}^n$, a set $Z \subseteq \mathbb{R}^n$, decide if $\exists T \geq 0, \ u : [0, T] \rightarrow U$ such that $x(T) \in Z$ where

$$x(0) = 0, \quad x'(t) = Ax(t) + u(t) \quad \text{for} \ t \in [0, T].$$
Study generalization:

**LTI Null-Set-Reachability problem**

Given a matrix $A \in \mathbb{Q}^{n \times n}$, a set of controls $U \subseteq \mathbb{R}^n$, a set $Z \subseteq \mathbb{R}^n$, decide if $\exists T \geq 0$, $u : [0, T] \rightarrow U$ such that $x(T) \in Z$ where

$$x(0) = 0, \quad x'(t) = Ax(t) + u(t) \quad \text{for } t \in [0, T].$$

This is trivially hard for $U = \{0\}$ and $Z = \{\text{hyperplane}\}$ because:
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**Continuous Skolem problem**

Given a matrix $A \in \mathbb{Q}^{n \times n}$ and $c, \; x_0 \in \mathbb{Q}^n$, decide if $\exists T \geq 0$ such that $c^T e^{At} x_0 = 0$. 


Hardness

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This is a well-known “hard” problem.
Taking $U = \{0\}$ is cheating:

- when $U = \{0\}$, reachable set is closed (or closed minus a point)
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- when $U = \{0\}$, reachable set is closed (or closed minus a point)

- when $U = B[-1,1]^m$, reachable set is open

This is completely different!
Our results: hardness

LTI Zonotope Null-Set-Reachability problem

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Theorem (Dantam, P.) The Continuous Nontangential Skolem problem reduces to this problem with a single input ($m = 1$), $A$ stable and $Z$ a hyperplane or a convex compact set of dimension $n - 1$. Continuous Nontangential Skolem problem

Given a matrix $A \in \mathbb{Q}^{n \times n}$ and $c, x_0 \in \mathbb{Q}^n$, decide if $\exists T \geq 0$ such that $f(t) = 0$ and $f'(t) \neq 0$ where $f(t) = c^T e^A t x_0 = 0$.

It is essentially as hard as the Continuous Skolem problem.
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It is essentially as hard as the Continuous Skolem problem.
Conclusion (continuous case)

LTI reachability problem: find $T$ and $u$ such that

$$x(0) = 0, \quad x'(t) = Ax(t) + Bu(t), \quad u(t) \in [-1, 1]^m$$

satisfies $x(T) = \text{target}$. Very natural problem in control theory.

---

Point reachability is

- decidable in dimension 2 or with spectral constraints,
- conditionally decidable with real eigenvalues,
- conditionally decidable in bounded time,

Set reachability is Nontangential Continuous Skolem hard.
The continuous case is much harder than expected. What about the discrete case?
The problem

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$LTI$–REACHABILITY is decidable if $U$ is an affine subspace of $\mathbb{R}^d$. 
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Almost no exact results for other classes of $U$ in particular when $U$ is bounded (which is the most natural case).
Theorem (Fijalkow, Ouaknine, P. Sousa-Pinto, Worrell)

\[ \text{LTI-REACHABILITY is} \quad \text{undecidable} \quad \text{if} \quad U \quad \text{is a finite union of affine subspaces}. \]

\[ \text{\quad Skolem-hard} \quad \text{if} \quad U = \{0\} \cup V \text{ where } V \text{ is an affine subspace} \]

\[ \text{\quad Positivity-hard} \quad \text{if} \quad U \text{ is a convex polytope} \]

Since we cannot solve Skolem/Positivity, we need some strong assumptions for decidability.
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\[\text{Theorem (Fijalkow, Ouaknine, P . Sousa-Pinto, Worrell)}\]
\[\text{LTI-REACHABILITY is decidable for simple systems.}\]

Remark: in fact we can decide reachability to a convex polytope \(Q\).

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**Remark:** in fact we can decide reachability to a convex polytope \(Q\).

Assumptions imply that the reachable set is an open convex bounded set, but not always a polytope!
Why is this problem hard

The reachable set $A^*(U)$ can have \textbf{infinitely} many faces.

\[
A = \begin{bmatrix}
\frac{1}{3} & 0 \\
0 & \frac{2}{3}
\end{bmatrix}
\]

\[
A^*(U)
\]
Why is this problem hard

The reachable set $A^*(U)$ can have **faces of lower dimension**: the "top" extreme point does not belong to any facet.

$$A^*(U)$$

$$A = \begin{bmatrix} \frac{2}{3} & 0 \\ 0 & \frac{1}{3} \end{bmatrix}$$

$(0, 2)$

$U$

$(-1, 0)$

$(1, 0)$
Why is this problem hard

**Approach:** two semi-decision procedures

- **reachability:** under-approximations of the reachable set
- **non-reachability:** **separating hyperplanes**
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**Approach:** two semi-decision procedures

- reachability: under-approximations of the reachable set
- non-reachability: **separating hyperplanes**

Further difficulty: a separating hyperplane may not be supported by a facet of either $A^*(U)$ or $Q$. 
Why is this problem hard

\[ B = \begin{bmatrix} \frac{2}{3} & \frac{1}{3} \\ \frac{1}{3} & 0 \end{bmatrix} \]

Even more difficulty: \( B^*(V) \) has two extreme points that do not belong to any facet and have rational coordinates, but whose (unique) separating hyperplane requires the use of algebraic irrationals.
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Even more difficulty: \( B^*(V) \) has two extreme points that do not belong to any facet and have rational coordinates, but whose (unique) separating hyperplane requires the use of algebraic irrationals.

**Theorem (Non-reachable instances)**

There is a separating hyperplane with algebraic coefficients.
Conclusion on control

Exact reachability for LTI systems:

- decidability crucially depends on the shape of the control set
- even with convex bounded inputs, the problem is very hard (Skolem/Positivity, open for 70 years)
- we can recover decidability using strong spectral assumptions
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Exact reachability for LTI systems:

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Despite an extensive literature in control theory, the decidability control problems is still very open.
Invariant Synthesis
Does this program halt?

Affine program

\[
\begin{align*}
x & := 2^{-10} \\
y & := 1 \\
\text{while } y \geq x \text{ do} \\
\begin{bmatrix} x \\ y \end{bmatrix} & := \begin{bmatrix} 2 & 0 \\ 7/4 & 1/4 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}
\end{align*}
\]
Does this program halt?

Affine program

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\]

Certificate of non-termination:

\[ x^2 y - x^3 = 1023 \]

\[ 1073741824 (2) \]

\( (2) \) is an invariant: it holds at every step.

\( (2) \) implies the guard is true.
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\[y \quad x\]

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80 / 101
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\((2)\) implies the **guard** is true.
invariant = overapproximation of the reachable states
Invariants

invariant = overapproximation of the reachable states

inductive invariant = invariant preserved by the transition relation
Inductive invariants: example
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$x, y, z$ range over $\mathbb{Q}$

$f_i : \mathbb{R}^3 \rightarrow \mathbb{R}^3$
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\[ S_1, S_2, S_3 \text{ are the reachable states} \]
Inductive invariants: example

$x, y, z$ range over $\mathbb{Q}$

$f_i : \mathbb{R}^3 \rightarrow \mathbb{R}^3$

$S_1, S_2, S_3$ is also an inductive invariant
Inductive invariants: example

$x, y, z$ range over $\mathbb{Q}$

$f_i : \mathbb{R}^3 \rightarrow \mathbb{R}^3$

$l_1, l_2, l_3$ is an invariant
Inductive invariants: example

$x, y, z$ range over $\mathbb{Q}$

$f_i : \mathbb{R}^3 \rightarrow \mathbb{R}^3$

$l_1, l_2, l_3$ is NOT an inductive invariant
Inductive invariants: example

$x, y, z$ range over $\mathbb{Q}$

$f_i : \mathbb{R}^3 \rightarrow \mathbb{R}^3$

$l_1, l_2, l_3$ is an inductive invariant
The classical approach to the verification of temporal safety properties of programs requires the construction of inductive invariants [...]. Automation of this construction is the main challenge in program verification.

D. Beyer, T. Henzinger, R. Majumdar, and A. Rybalchenko
*Invariant Synthesis for Combined Theories*, 2007
Which invariants?

- Octagons
- Polyhedrons
- Semialgebraic sets
- Intervals
- Affine/linear sets
- Algebraic sets = polynomial equalities
Affine programs

- Nondeterministic branching (no guards)
- All assignments are affine
- Allow nondeterministic assignments (x := *)

- Can overapproximate complex programs
- Covers existing formalisms: probabilistic, quantum, quantitative automata
Affine programs

- Nondeterministic branching (no guards)

![Diagram of nondeterministic branching](image)

- Can overapproximate complex programs
- Covers existing formalisms: probabilistic, quantum, quantitative automata
Affine programs

- Nondeterministic branching (no guards)
- All assignments are affine

\[
x := 3x - 7y + 1
\]
Affine programs

- Nondeterministic branching (no guards)
- All assignments are affine
- Allow nondeterministic assignments ($x := \ast$)
Affine programs

- Nondeterministic branching (no guards)
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- Allow nondeterministic assignments ($x := \ast$)

$x := 3x - 7y + 1$

- Can overapproximate complex programs
Affine programs

- Nondeterministic branching (no guards)
- All assignments are affine
- Allow nondeterministic assignments ($x := \ast$)

Can overapproximate complex programs

Covers existing formalisms: probabilistic, quantum, quantitative automata
Affine Relationships Among Variables of a Program*

Michael Karr

Received May 8, 1974

Summary. Several optimizations of programs can be performed when in certain regions of a program equality relationships hold between a linear combination of the variables of the program and a constant. This paper presents a practical approach to detecting these relationships by considering the problem from the viewpoint of linear algebra. Key to the practicality of this approach is an algorithm for the calculation of the "sum" of linear subspaces.

Theorem (Karr 76)

There is an algorithm which computes, for any given affine program over \(\mathbb{Q}\), its strongest affine inductive invariant.
Discovering Affine Equalities Using Random Interpretation

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ABSTRACT
We present a new polynomial-time randomized algorithm for discovering affine equalities involving variables in a program.

Keywords
Affine Relationships, Linear Equalities, Random Interpretation, Randomized Algorithm
A Note on Karr’s Algorithm

Markus Müller-Olm\textsuperscript{1,*} and Helmut Seidl\textsuperscript{2}

Abstract. We give a simple formulation of Karr’s algorithm for computing all affine relationships in affine programs. This simplified algorithm runs in time $O(nk^3)$ where $n$ is the program size and $k$ is the number of program variables assuming unit cost for arithmetic operations. This improves upon the original formulation by a factor of $k$. Moreover, our re-formulation avoids exponential growth of the lengths of intermediately occurring numbers (in binary representation) and uses less complicated elementary operations. We also describe a generalization that determines all polynomial relations up to degree $d$ in time $O(nk^{3d})$.

Theorem (ICALP 2004)

There is an algorithm which computes, for any given affine program over $\mathbb{Q}$, all its polynomial inductive invariants up to any fixed degree $d$. 
Why fixed degree is not enough

Paraboloid
\[ z = x^2 + y^2 \]

Union of 3 hyperplanes
\[(x - y)(10y + x)(y + 10x) = 0 \]
Why fixed degree is not enough

- Paraboloid

\[ z = x^2 + y^2 \]
Why fixed degree is not enough

- Paraboloid

\[ z = x^2 + y^2 \]
Why fixed degree is not enough

- Paraboloid
- Union of 3 hyperplanes

\[ z = x^2 + y^2 \]
\[ (x - y)(10y + x)(y + 10x) = 0 \]
Why fixed degree is not enough

- Paraboloid
  \[ z = x^2 + y^2 \]

- Union of 3 hyperplanes
  \[(x - y)(10y + x)(y + 10x) = 0\]
Why fixed degree is not enough

- Paraboloid
  \[ z = x^2 + y^2 \]
- Union of 3 hyperplanes
  \[ (x - y)(10y + x)(y + 10x) = 0 \]
Main result

Theorem (Hrushovski, Ouaknine, P., Worrell, 2018)

There is an algorithm which computes, for any given affine program over $\overline{\mathbb{Q}}$, its strongest polynomial inductive invariant.
Theorem (Hrushovski, Ouaknine, P., Worrell, 2018)

There is an algorithm which computes, for any given affine program over $\mathbb{Q}$, its strongest polynomial inductive invariant.

- strongest polynomial invariant $\iff$ smallest algebraic set
Theorem (Hrushovski, Ouaknine, P., Worrell, 2018)

There is an algorithm which computes, for any given affine program over $\overline{\mathbb{Q}}$, its strongest polynomial inductive invariant.

- strongest polynomial invariant $\iff$ smallest algebraic set
- Thus our algorithm computes all polynomial relations that always hold among program variables at each program location, in all possible executions of the program
Main result

Theorem (Hrushovski, Ouaknine, P., Worrell, 2018)

There is an algorithm which computes, for any given affine program over $\mathbb{Q}$, its strongest polynomial inductive invariant.

- strongest polynomial invariant $\iff$ smallest algebraic set
- Thus our algorithm computes all polynomial relations that always hold among program variables at each program location, in all possible executions of the program
- We represent this using a finite basis of polynomial equalities
At the edge of decidability

\[ x := M_1 x \]
\[ x := M_2 x \]
\[ x := M_0 x \]
\[ \ldots \]
\[ x := M_k x \]

Theorem (Markov 1947)
There is a fixed set of \(6 \times 6\) integer matrices \(M_1, \ldots, M_k\) such that the reachability problem "\(y\) is reachable from \(x_0\)" is undecidable.

Theorem (Paterson 1970)
The mortality problem "\(0\) is reachable from \(x_0\) with \(M_1, \ldots, M_k\)" is undecidable for \(3 \times 3\) matrices.

§ Original theorems about semigroups, reformulated with affine programs.
At the edge of decidability

\[ x := M_1 x \]
\[ x := M_2 x \]
\[ x := M_3 x \]
\[ \ldots \]
\[ x := M_k x \]

Theorem (Markov 1947§)

There is a fixed set of $6 \times 6$ integer matrices $M_1, \ldots, M_k$ such that the reachability problem “$y$ is reachable from $x_0$?” is undecidable.

---

§Original theorems about semigroups, reformulated with affine programs.
At the edge of decidability

\[
\begin{align*}
x &:= M_1 x \\
x &:= x_0 \\
x &:= M_2 x \\
& \quad \ldots \\
x &:= M_k x
\end{align*}
\]

**Theorem (Markov 1947\(^\S\))**

*There is a fixed set of \(6 \times 6\) integer matrices \(M_1, \ldots, M_k\) such that the reachability problem “\(y\) is reachable from \(x_0\)?” is undecidable.*

**Theorem (Paterson 1970\(^*\))**

*The mortality problem “\(0\) is reachable from \(x_0\) with \(M_1, \ldots, M_k\)?” is undecidable for \(3 \times 3\) matrices.*

\(^\S\)Original theorems about semigroups, reformulated with affine programs.
Our algorithm relies on this result:

**Theorem (Derksen, Jeandel and Koiran, 2004)**

_There is an algorithm which computes, for any given affine program over \( \mathbb{Q} \) using only invertible transformations, its strongest polynomial inductive invariant._

**Equivalently,** compute the Zariski closure of a finitely generated groups of matrices.
From groups to semigroup

Theorem (Hrushovski, Ouaknine, P., Worrell, 2018)

There is an algorithm that computes the Zariski closure of any finitely semigroup of matrices (with algebraic coefficients), given its generators as inputs.

Corollary

Given an affine program, we can compute for each location the ideal of all polynomial relations that hold at that location.
Going hybrid: a bouncing ball

\[ \dot{x} = v_x \]
\[ \dot{y} = v_y \]
\[ \dot{v}_x = 0 \]
\[ \dot{v}_y = -g \]
\[ \dot{t} = 1 \]
\[ x(0) = 0 \]
\[ y(0) = h \]
\[ v_x(0) = c \]
\[ v_y(0) = 0 \]

Invariants:
\[ v_x = c \]
\[ x = t c \]
\[ v_y^2 y + 2 g (y - h) = 0 \]

recover conservation of energy!
Going hybrid: a bouncing ball

Invariants:

- $v_x := c$
- $v_y := 0$
- $\dot{x} = v_x$
- $\dot{y} = v_y$
- $\dot{v}_x = 0$
- $\dot{v}_y = -g$
- $t = 1$

$\triangleright$ affine program: collision
+ linear differential equation: mechanics
= linear hybrid automaton

$\dot{x} = v_x$
$\dot{y} = v_y$
$\dot{v}_x = 0$
$\dot{v}_y = -g$
$t = 1$
Going hybrid: a bouncing ball

\[\begin{align*}
\dot{x} &= v_x \\
\dot{y} &= v_y \\
\dot{v}_x &= 0 \\
\dot{v}_y &= -g \\
\dot{t} &= 1
\end{align*}\]

- Invariants:
  - \(v_x = c\)
  - \(x = tc\)
  - \(v_y^2 + 2g(y - h) = 0\)

- affine program: collision
- linear differential equation: mechanics

= linear hybrid automaton

recover conservation of energy!
Example: RC circuit

\[ I \quad \text{CLOSED} \quad I_R \quad V_R \quad R \quad Q \quad V_C \]

\[ \dot{I} = 0 \]

\[ \dot{I}_R = -\frac{1}{RC} I \]

\[ \dot{V}_R = -\frac{1}{C} I \]

\[ \dot{Q} = I \quad \dot{V}_C = \frac{1}{C} I \]

\[ I = \frac{1}{R} (V - V_C) \]

\[ I_R = -\frac{1}{R} V \]

\[ V = V - V_C \]

\[ t \]
Example: RC circuit

\[ I \begin{array}{c} \text{CLOSED} \\ \text{OPEN} \end{array} \quad I_R \quad V_R \quad R \quad Q \quad V_C \]

\[ \begin{align*}
\dot{I} &= 0 \\
\dot{I}_R &= -\frac{1}{RC} I_R \\
\dot{V}_R &= -\frac{1}{C} I_R \\
\dot{Q} &= I_R \\
\dot{V}_C &= \frac{1}{C} I_R 
\end{align*} \]
Example: RC circuit

\[
\begin{align*}
\dot{I} &= 0 \\
\dot{I}_R &= -\frac{1}{RC} I_R \\
\dot{V}_R &= -\frac{1}{C} I_R \\
Q &= I_R \\
\dot{V}_C &= \frac{1}{C} I_R
\end{align*}
\]
Example: RC circuit

\[ I = 0 \]
\[ \dot{I}_R = -\frac{1}{RC} I_R \]
\[ \dot{V}_R = -\frac{1}{C} \dot{I}_R \]
\[ Q = I_R \]
\[ \dot{V}_C = \frac{1}{C} I_R \]

\[ I = \frac{1}{R} (V - V_C) \]
\[ \dot{I}_R = \frac{1}{R} (V - V_C) \]
\[ \dot{V}_R = V - V_C \]

\[ \dot{I} = -\frac{1}{RC} I_R \]
\[ \dot{I}_R = -\frac{1}{RC} I_R \]
\[ \dot{V}_R = -\frac{1}{C} \dot{I}_R \]
\[ Q = I_R \]
\[ \dot{V}_C = \frac{1}{C} I_R \]
Example: RC circuit

![Diagram of an RC circuit with open and closed switch states.]

Invariants

**OPEN**
- \( Q = CV_C \)
- \( V_R = RI_R \)
- \( I = 0 \)
- \( V_R = -V_C \)

**CLOSED**
- \( Q = CV_C \)
- \( V_R = RI_R \)
- \( I = I_R \)
- \( V_R = V - V_C \)

**OPEN**
- \( \dot{I} = 0 \)
- \( \dot{I}_R = -\frac{1}{RC}I_R \)
- \( \dot{V}_R = -\frac{1}{C}I_R \)
- \( \dot{Q} = I_R \)
- \( \dot{V}_C = \frac{1}{C}I_R \)

**CLOSED**
- \( \dot{I} = \frac{1}{R}(V - V_C) \)
- \( \dot{I}_R = \frac{1}{R}(V - V_C) \)
- \( \dot{V}_R = V - V_C \)
- \( \dot{I} = -\frac{1}{RC}I_R \)
- \( \dot{I}_R = -\frac{1}{RC}I_R \)
- \( \dot{V}_R = -\frac{1}{C}I_R \)
- \( \dot{Q} = I_R \)
- \( \dot{V}_C = \frac{1}{C}I_R \)
Linear Hybrid Automata

- Nondeterministic branching (no guards)
- All assignments are affine
- **Linear differential equations** in each location

\[
\dot{x} = 2y - x \\
\dot{y} = x - y \\
x := 3x - 7y + 1
\]

\[
\dot{X} = AX \\
\dot{X} = BX
\]
Linear Hybrid Automata

- Nondeterministic branching (no guards)
- All assignments are affine
- Linear differential equations in each location

More general than affine programs
More general than linear differential equations
Theorem (Majumdar, Ouaknine, P., Worrell, 2020)

*There is an algorithm that computes, for any given guard-free linear hybrid automaton over \( \mathbb{Q} \), its strongest polynomial inductive invariant.*
From affine programs to hybrid automata

Theorem (Majumdar, Ouaknine, P., Worrell, 2020)

There is an algorithm that computes, for any given guard-free linear hybrid automaton over $\mathbb{Q}$, its strongest polynomial inductive invariant.

For systems with purely continuous dynamics, i.e. no discrete transitions, called switching systems:

Theorem (Hrushovski, Ouaknine, P., Worrell, 2018)

There is no algorithm that computes the strongest algebraic inductive invariant for the class of switching systems with equality guards.
**Theorem (Majumdar, Ouaknine, P., Worrell, 2020)**

There is an algorithm that computes, for any given guard-free linear hybrid automaton over \( \mathbb{Q} \), an affine program over \( \mathbb{Q} \) that has the same polynomial inductive invariants.
Theorem (Majumdar, Ouaknine, P., Worrell, 2020)

There is an algorithm that computes, for any given guard-free linear hybrid automaton over \( \mathbb{Q} \), an affine program over \( \mathbb{Q} \) that has the same polynomial inductive invariants.

\[
\begin{align*}
\dot{x} &= v_x \\
\dot{y} &= v_y \\
\dot{v}_x &= 0 \\
\dot{v}_y &= -g \\
t &= 1 \\
v_y &= -v_y \\
t &= 0 \\
x &= 0 \\
y &= h \\
v_x &= c \\
v_y &= 0
\end{align*}
\]
Theorem (Majumdar, Ouaknine, P., Worrell, 2020)

There is an algorithm that computes, for any given guard-free linear hybrid automaton over $\mathbb{Q}$, an affine program over $\mathbb{Q}$ that has the same polynomial inductive invariants.
Linear Differential Equations

For $x(t) \in \mathbb{R}^n$ and $A$ a rational matrix, consider

$$\dot{x} = Ax$$

The solution is

$$x(t) = e^{At} x(0)$$

where $e^X$ is the matrix exponential.
Linear Differential Equations

For \( x(t) \in \mathbb{R}^n \) and \( A \) a rational matrix, consider

\[
\dot{x} = Ax
\]

The solution is

\[
x(t) = e^{At}x(0)
\]

where \( e^X \) is the matrix exponential. Recall that:

- strongest algebraic invariant = smallest algebraic set
- smallest algebraic set containing \( X = \) Zariski closure \( \overline{X} \) of \( X \)

Lemma

Let \( A \) be a rational matrix, there exists \( B \) an algebraic matrix such that

\[
\langle B \rangle = \langle e^A \rangle = \{ e^{At} : t \in \mathbb{R} \}.
\]
For \( x(t) \in \mathbb{R}^n \) and \( A \) a rational matrix, consider

\[
\dot{x} = Ax
\]

The solution is

\[
x(t) = e^{At}x(0)
\]

where \( e^X \) is the matrix exponential. Recall that:

- strongest algebraic invariant = smallest algebraic set
- smallest algebraic set containing \( X = \text{Zariski closure} \overline{X} \) of \( X \)

**Lemma**

Let \( A \) be a rational matrix, there exists \( B \) an algebraic matrix such that

\[
\langle B \rangle = \langle e^A \rangle = \{ e^{At} : t \in \mathbb{R} \}.
\]

- obvious candidate \( B = e^A \) is not algebraic
- “reverse-engineer” \( B \) algebraic to encode some multiplicative relations between the eigenvalues
Complexity of computing the Zariski closure

How expensive is it to compute this strongest invariant?

```
linear hybrid reduce semigroup reduce group
```
Complexity of computing the Zariski closure

How expensive is it to compute this strongest invariant?

Theorem (Derksen, Jeandel and Koiran, 2004)

There is an algorithm that computes the Zariski closure of any finitely group of matrices, given its generators as inputs.

No complexity bounds. It is not clear it is even elementary.
Complexity of computing the Zariski closure

How expensive is it to compute this strongest invariant?

Theorem (Derksen, Jeandel and Koiran, 2004)

\[ \text{There is an algorithm that computes the Zariski closure of any finitely group of matrices, given its generators as inputs.} \]

No complexity bounds. It is not clear it is even elementary.

Theorem (Nosan, P., Schmitz, Shirmohammadi, Worrell, 2022)

\[ \text{Given a finite set } S \text{ of invertible matrices of dimension } n, \text{ the algebraic group } G := \langle S \rangle \text{ can be defined with equations of degree at most septuply exponential in } n. \]
Summary

- invariant = overapproximation of reachable states
- invariants allow verification of safety properties
- guard-free linear hybrid automata:
  - nondeterministic branching, no guards, affine assignments
  - linear differential equations

Theorem (Majumdar, Ouaknine, P., Worrell, 2020)

There is an algorithm that computes, for any given guard-free linear hybrid automaton over \( \mathbb{Q} \), its strongest polynomial inductive invariant.