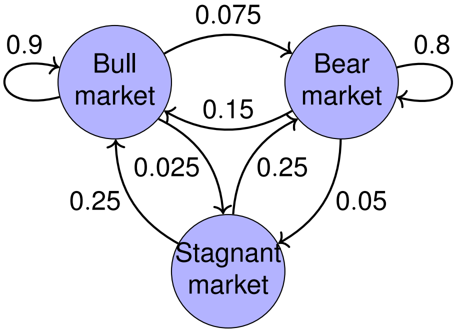
Linear Dynamical Systems

Amaury Pouly

Overview

Examples: while loop, Markov chain



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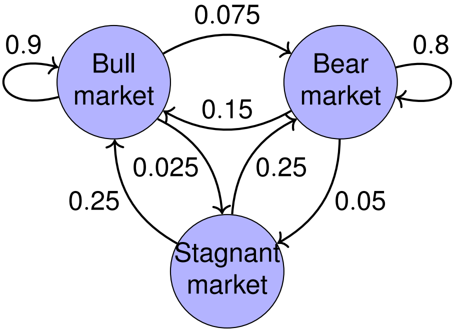

State: $X = (p_{bull}, p_{bear}, p_{stag}) \in [0,1]^3$
Transitions:

$$A = \begin{bmatrix} 0.9 & 0.15 & 0.25 \\ 0.075 & 0.8 & 0.25 \\ 0.025 & 0.05 & 0.5 \end{bmatrix}$$

$\rightarrow$ Linear dynamical system

$$X_{n+1} = AX_n$$

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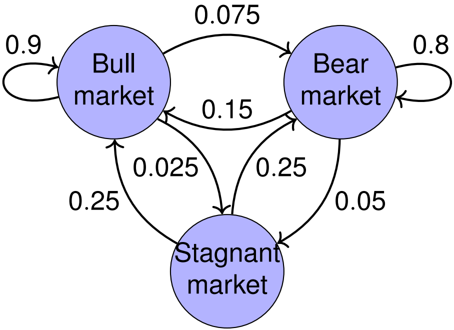

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Linear loop

$p_{bull} := 0$
$p_{bear} := 1$
$p_{stag} := 0$
while $p_{bull} \leqslant 1/2$ do

$$\begin{bmatrix} p_{bull} \\ p_{bear} \\ p_{stag} \end{bmatrix} := A \begin{bmatrix} p_{bull} \\ p_{bear} \\ p_{stag} \end{bmatrix}$$

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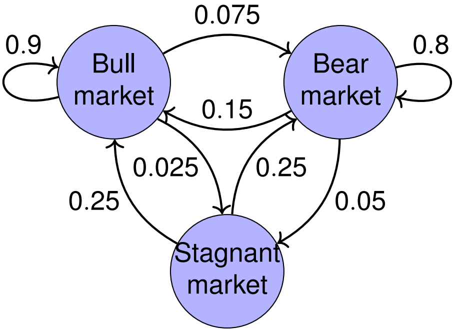

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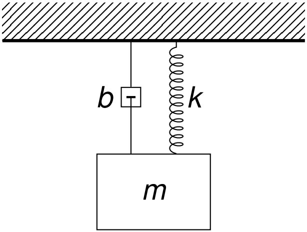
The loop terminates if and only if the probability of a bull market is $> 1/2$.

Example: mass-spring-damper system



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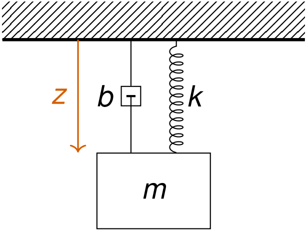

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Equation of motion:

$$mz'' = -kz - bz' + mg$$

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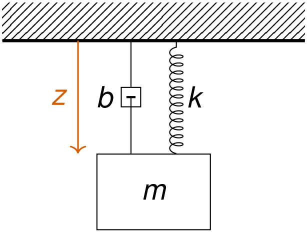

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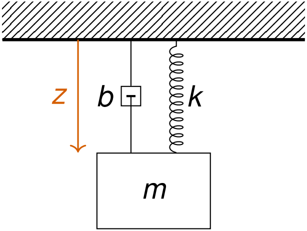

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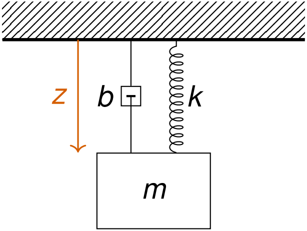

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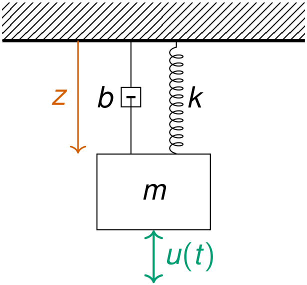

with external input $u(t)$.

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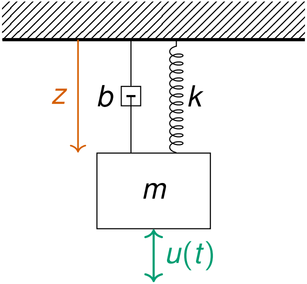

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$$X' = AX + Bu$$

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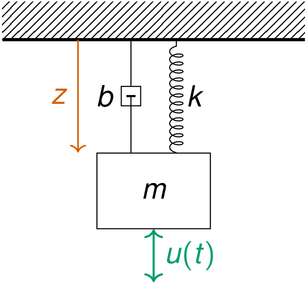

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Can be used to model a car suspension.

Linear dynamical systems

Discrete case

$$x(n+1) = Ax(n)$$

- biology,
- software verification,
- probabilistic model checking,
- combinatorics,
- ....

Continuous case

$$x'(t) = Ax(t)$$

- biology,
- physics,
- probabilistic model checking,
- electrical circuits,
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Typical questions

- reachability
- safety

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- reachability
- safety
- controllability
- optimal control
- feedback control
- ...

More complicated programs

Linear loop with if

$x := 2^{-10}$
$y := 1$
while $y \geqslant x$ do
    if $y \geqslant 2x$ then
$$\begin{bmatrix} x \\ y \end{bmatrix} := \begin{bmatrix} 2 & 0 \\ 1 & 4 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$
    else
$$\begin{bmatrix} x \\ y \end{bmatrix} := \begin{bmatrix} 2 & 3 \\ -3 & 7 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

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- invariant* synthesis also hard

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Overapproximate behaviours

- reachability still undecidable
- invariant synthesis possible

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Does this program halt?

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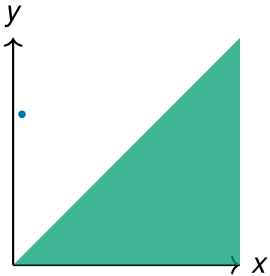

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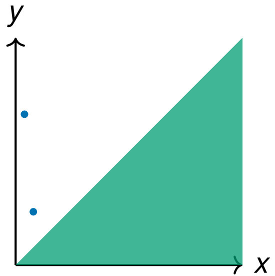

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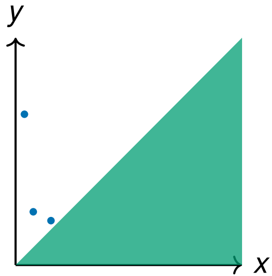

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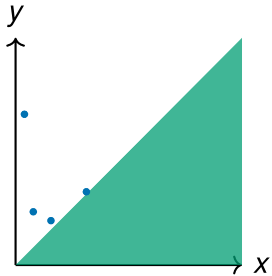

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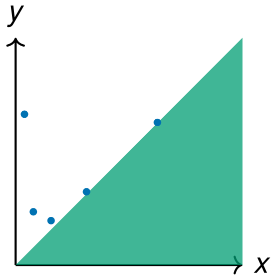

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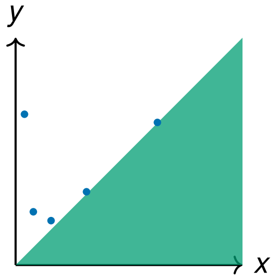

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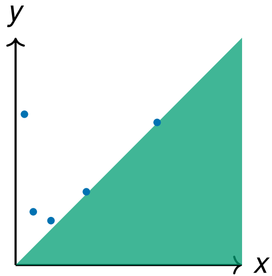

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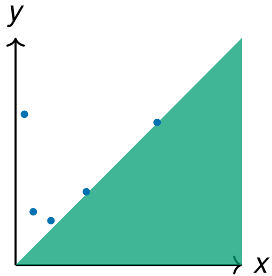
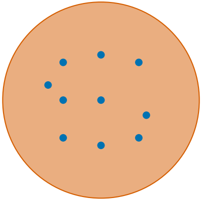


- (2) is an invariant: it holds at every step
- (2) implies the guard is true

Invariants

invariant = overapproximation of the reachable states



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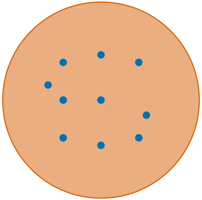
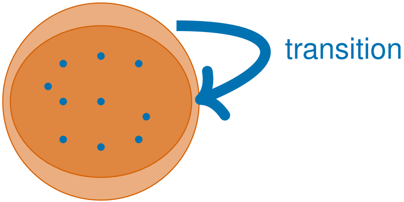
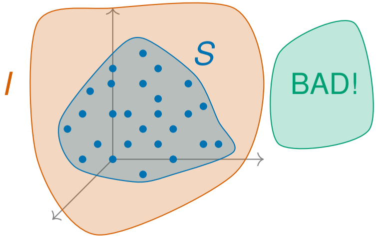


inductive invariant = invariant preserved by the transition relation



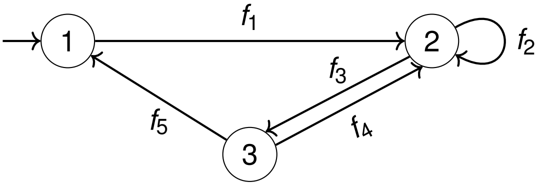
Why Invariants?



The classical approach to the verification of temporal safety properties of programs requires the construction of ***inductive invariants*** *[...].* ***Automation of this construction is the main challenge in program verification.***

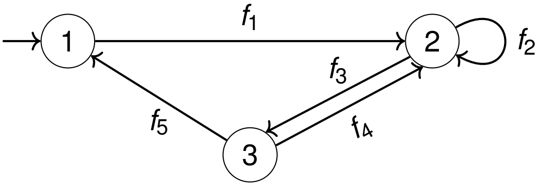
D. Beyer, T. Henzinger, R. Majumdar, and A. Rybalchenko
Invariant Synthesis for Combined Theories, 2007

Affine programs



Affine programs

- Nondeterministic branching (no guards)



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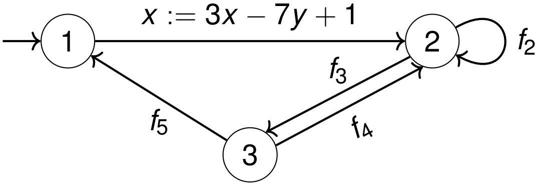

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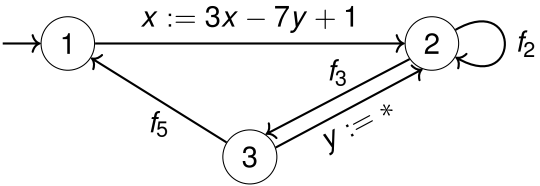

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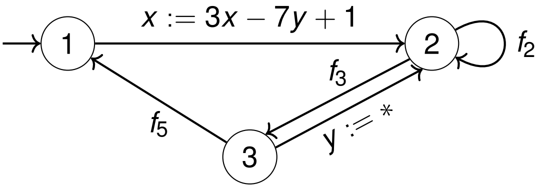
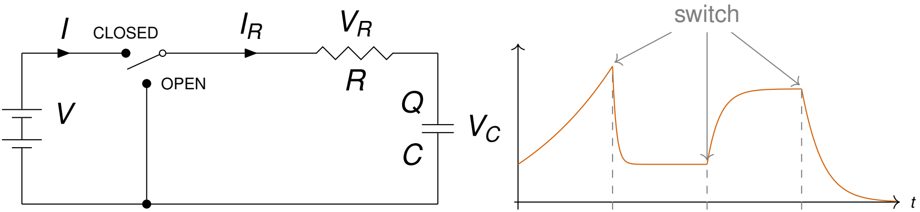


- Can overapproximate complex programs
- Covers existing formalisms:
  finite, probabilistic, quantum, quantitative automata

RC circuit



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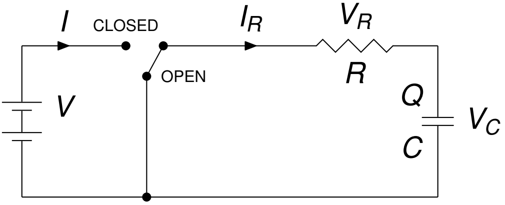
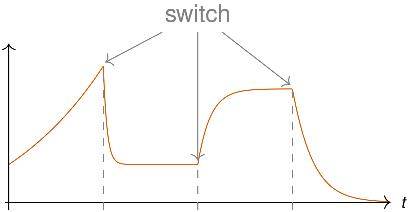
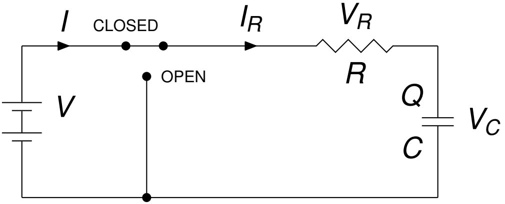
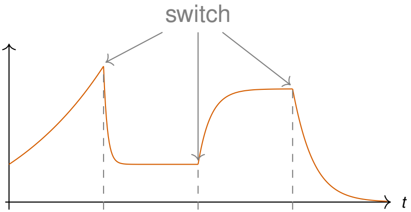




OPEN

$$\boxed{\begin{aligned} \dot{I} &= 0 \\ \dot{I}_R &= -\frac{1}{RC} I_R \\ \dot{V}_R &= -\frac{1}{C} I_R \\ \dot{Q} &= I_R \\ \dot{V}_C &= \frac{1}{C} I_R \end{aligned}}$$





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CLOSED

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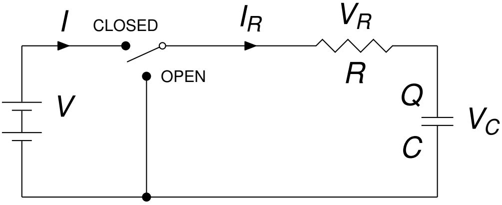
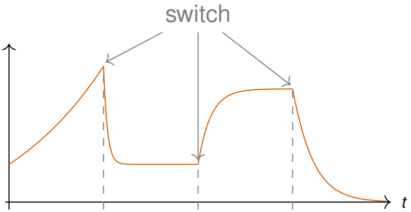
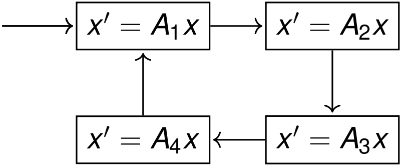
RC circuit





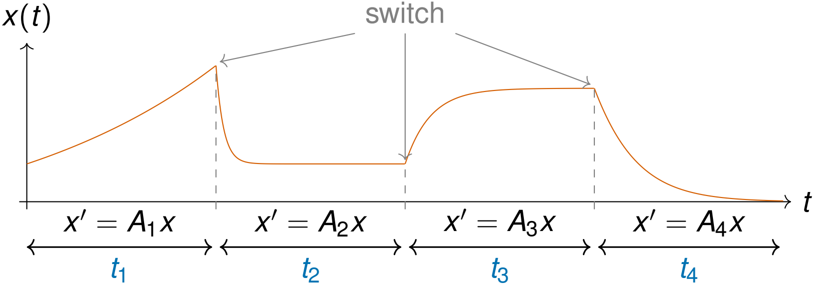
OPEN		CLOSED
$\dot I = 0$ $\dot I_R = -\frac{1}{RC} I_R$ $\dot V_R = -\frac{1}{C} I_R$ $\dot Q = I_R$ $\dot V_C = \frac{1}{C} I_R$	OPEN → CLOSED: $I := \frac{1}{R}(V - V_C)$, $I_R := \frac{1}{R}(V - V_C)$, $V_R := V - V_C$ CLOSED → OPEN: $I := 0$, $I_R := -\frac{1}{R} V_C$, $V_R := -V_C$	$\dot I = -\frac{1}{RC} I_R$ $\dot I_R = -\frac{1}{RC} I_R$ $\dot V_R = -\frac{1}{C} I_R$ $\dot Q = I_R$ $\dot V_C = \frac{1}{C} I_R$

Switching systems



Restricted hybrid system:

- linear dynamics
- no guards (nondeterministic)
- no discrete updates



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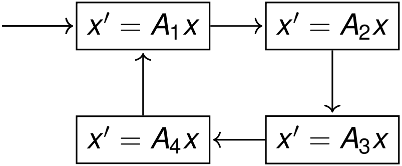

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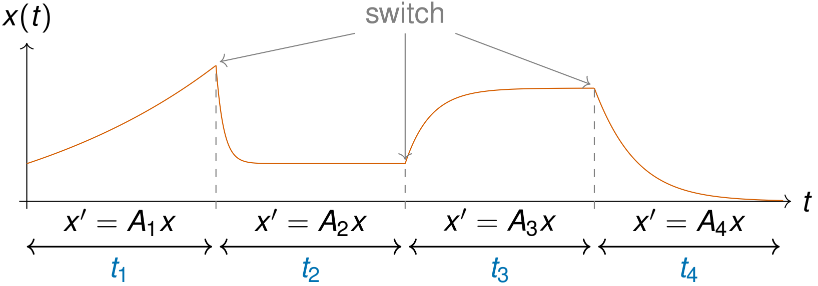
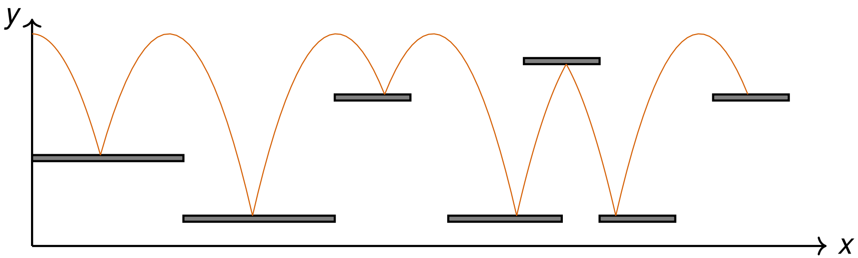


- reachability also undecidable
- invariant synthesis possible

Going hybrid: a bouncing ball



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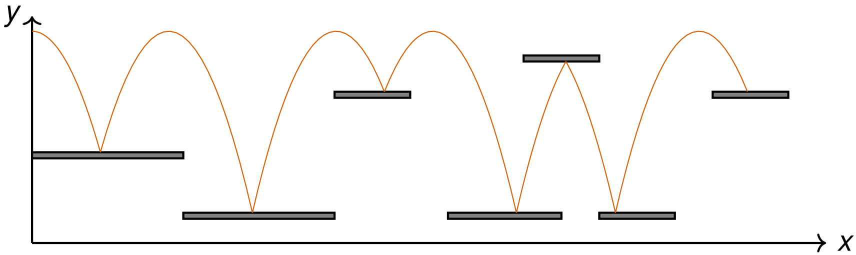

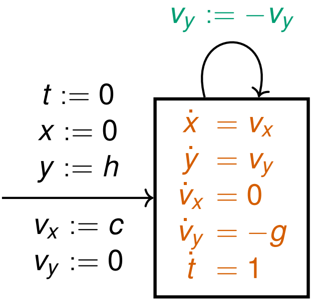

- affine program: collision
- \+ linear differential equation: mechanics
- = linear hybrid automaton

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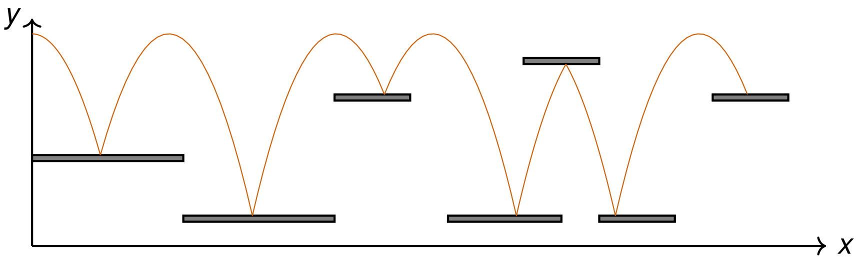


$$v_y := -v_y$$

$$t := 0 \quad x := 0 \quad y := h \quad v_x := c \quad v_y := 0$$

$$\begin{aligned} \dot{x} &= v_x \\ \dot{y} &= v_y \\ \dot{v}_x &= 0 \\ \dot{v}_y &= -g \\ \dot{t} &= 1 \end{aligned}$$

- affine program: collision
- \+ linear differential equation: mechanics
- = linear hybrid automaton

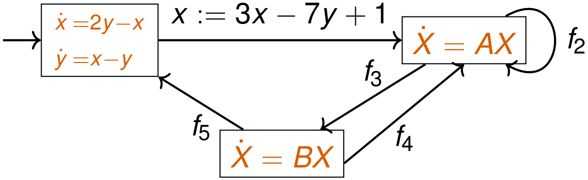
Invariants:

- $v_x = c$
- $x = tc$
- $v_y^2 + 2g(y - h) = 0$

recover conservation of energy!

Linear Hybrid Automata

- Nondeterministic branching (no guards)
- All assignments are affine
- Linear differential equations in each location



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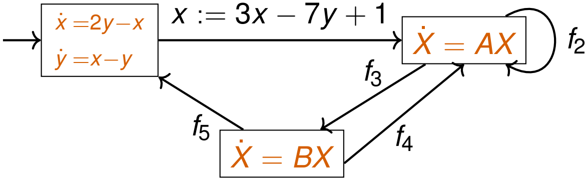
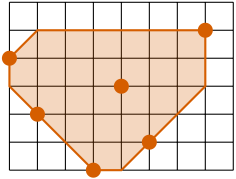


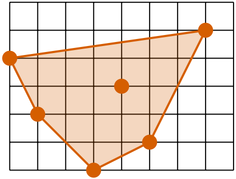
- More general than affine programs
- More general than linear differential equations

Which invariants?



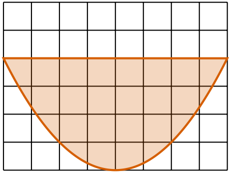
Octagons

$\leqslant$



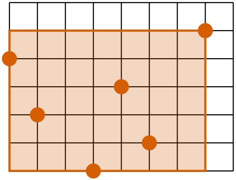
Polyhedrons

$\leqslant$

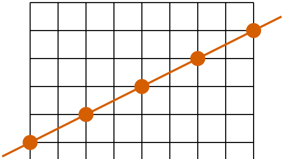


Semialgebraic sets

$\rotatebox{90}{$\leqslant$}$ $\rotatebox{90}{$\leqslant$}$ $\rotatebox{90}{$\leqslant$}$

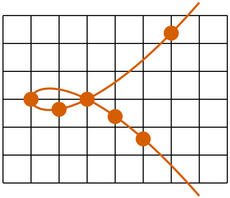


Intervals



Affine/linear sets

$\leqslant$



Algebraic sets =
polynomial equalities

Linear system with rounding

Rounding: $\lfloor \cdot \rceil$ = round to nearest integer

$$A = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \in \mathbb{Q}^{2\times 2}, \qquad \left\lfloor \begin{pmatrix} x \\ y \end{pmatrix} \right\rceil = \begin{pmatrix} \lfloor x \rceil \\ \lfloor y \rceil \end{pmatrix}$$

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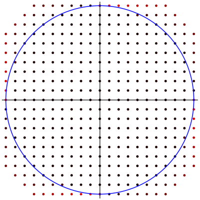

$r = 10, \theta = \pi/42$

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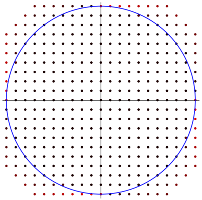

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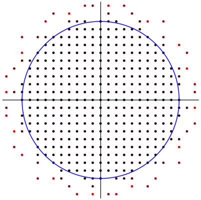

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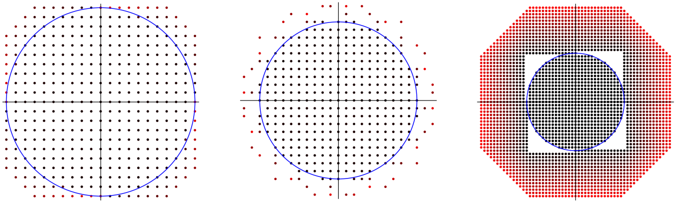

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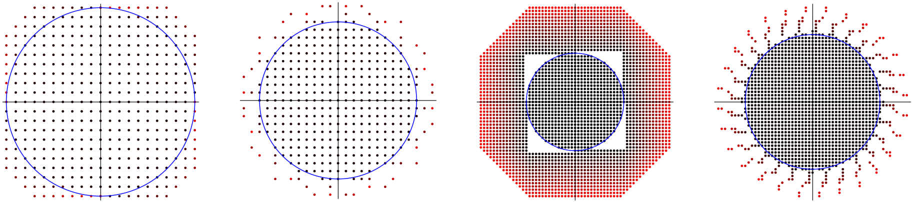

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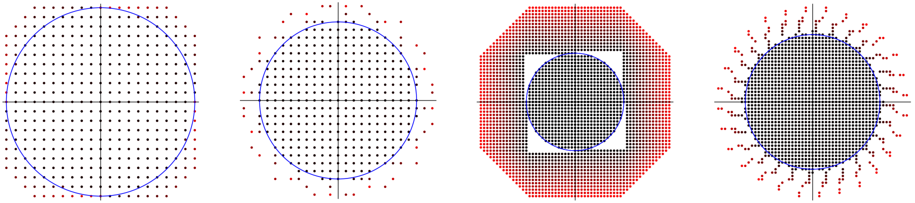

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Open problems! Only known for a few specific values of θ.

Linear dynamical systems are ubiquitous...

... and lead to very interesting mathematics!

Interesting related mathematics

- Linear recurrent sequences (LRS)

$$x_{n+k} = a_{k-1}a_{n+k-1} + \cdots + x_0 x_n$$

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$$f(t) = P_1(t) e^{\lambda_1 t} + \cdots + P_n(t) e^{\lambda_n t}$$

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Reachability often harder/reduces to of these problems!

Algebraic numbers and conjectures

Algebraic number: root of polynomial with integer coefficients
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If a, b are algebraic numbers with $a \neq 0, 1$ and b irrational, then (any value of) a^b transcendental.

Example: $2^{\sqrt{2}}$ is transcendental.

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Why is this related to reachability?

- target is usually rational/algebraic
- reachability creates constraints between numbers

Example: given $a, b \in \mathbb{Q}$, $P \in \mathbb{Q}[X]$ polynomial, find t such that

$$P(t) = a \quad \text{and} \quad e^t = b$$

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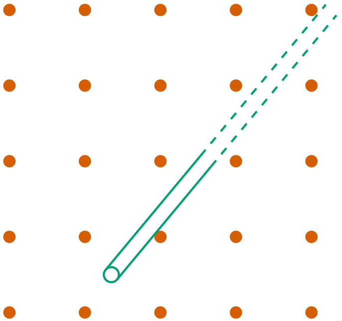
Biggest open question in this field: Schanuel's conjecture

Transcendental number theory

Many problems boil down to diophantine equations/approximations:

- Finding integer points in cones: Kronecker's theorem



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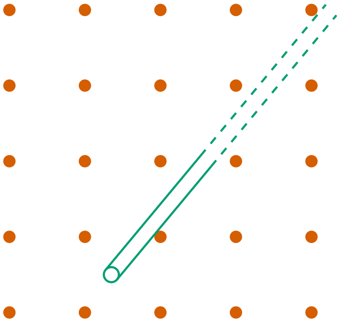

- Compare linear forms in logarithms: Baker's theorem

$$\sqrt{2} + \log\sqrt{3} - 3\log\sqrt{7} \overset{?}{=} 1 + \log 9 - \log \sqrt[42]{666}$$

(Semi-)group theory

Finitely generated matrix semigroup:

$A_1, \ldots, A_k \in \mathbb{Q}^{n \times n}$ generate a semigroup $S = \langle A_1, \ldots, A_k \rangle$

Example: $\mathsf{SL}_2(\mathbb{Z}) = \left\langle \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & -1 \\ 1 & 1 \end{bmatrix} \right\rangle$

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Undecidable in general, many decidable subclasses are known.
Equivalent to reachability of affine programs.

Algebraic geometry

Study systems of multivariate polynomial equations using abstract algebraic techniques, with applications to geometry.

Examples

$$\begin{aligned} x^2 + y^2 + z^2 - 1 = 0 \quad &\rightsquigarrow \quad \text{sphere in } \mathbb{R}^3 \\ x^2 + y^2 + z^2 = 1 \ \wedge \ x + y + z = 1 \quad &\rightsquigarrow \quad \text{"sliced" sphere in } \mathbb{R}^3 \\ x^2 + 1 = 0 \quad &\rightsquigarrow \quad \varnothing \text{ in } \mathbb{R} \\ x^2 + 1 = 0 \quad &\rightsquigarrow \quad \{i, -i\} \text{ in } \mathbb{C} \end{aligned}$$

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x^2+1=0 \quad &\rightsquigarrow \quad \{i,-i\} \text{ in } \mathbb{C}
\end{aligned}
$$

The field $\mathbb{K}$ is very important:

- real algebraic geometry: more “intuitive” but more difficult, really requires the study of *semi-algebraic sets*
- mainstream algebraic geometry: $\mathbb{K}$ is algebraically closed†, e.g. $\mathbb{C}$

† $\mathbb{K}$ is algebraically closed if every non-constant polynomial has a root in $\mathbb{K}$.

First-order theory of the reals

Many questions expressible in first-order logical theories:

- $\mathfrak{R}_0 = (\mathbb{R}, 0, 1, <, +, \cdot)$: decidable

$$\forall x, y \in \mathbb{R} \; \frac{x+y}{2} \geqslant \sqrt{xy}$$

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- $\mathfrak{R}_{\exp} = (\mathbb{R}, 0, 1, <, +, \cdot, \exp, \cos\restriction_{[0,1]})$: decidable subject to Schanuel's conjecture

$$\forall x \in \mathbb{R}\ x \neq 0 \Rightarrow t + te^{t} - 43e^{3t} \neq 1$$

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- Presburger arithmetic $(\mathbb{N}, 0, 1, <, +)$: decidable

$$\exists x \in \mathbb{N}^n\ Ax \geqslant b$$

Summary

Linear dynamical systems are ubiquitous and exact reachability questions lead to very interesting mathematical and logical questions.

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But...

- some systems are fundamentally nonlinear

$$x_{n+1} = x_n^2$$

Summary

Linear dynamical systems are ubiquitous and exact reachability questions lead to very interesting mathematical and logical questions.

But...

- some systems are fundamentally nonlinear

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- real programs manipulate data structures:

trees, arrays, ...

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- real programs manipulate data structures:

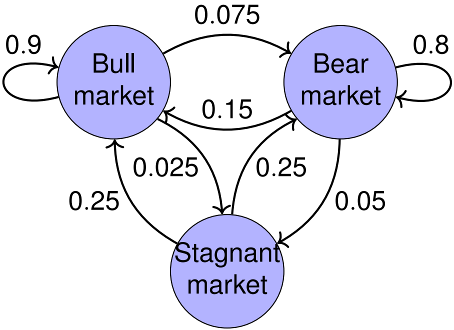
  trees, arrays, ...
- some programs are not sequential / nondeterministic

  probabilistic, concurrent/parallell, ...
- exact reachability is not the only approach

  testing, probabilistic model checking, incomplete algorithms, ...

Reachability

Examples: while loop, Markov chain



State: $X = (p_{bull}, p_{bear}, p_{stag}) \in [0,1]^3$
Transitions:

$$A = \begin{bmatrix} 0.9 & 0.15 & 0.25 \\ 0.075 & 0.8 & 0.25 \\ 0.025 & 0.05 & 0.5 \end{bmatrix}$$

$\to$ Linear dynamical system

$$X_{n+1} = AX_n$$

Linear loop

$p_{bull} := 0$
$p_{bear} := 1$
$p_{stag} := 0$
while $p_{bull} \leqslant 1/2$ do

$$\begin{bmatrix} p_{bull} \\ p_{bear} \\ p_{stag} \end{bmatrix} := A \begin{bmatrix} p_{bull} \\ p_{bear} \\ p_{stag} \end{bmatrix}$$

The loop terminates if and only if the probability of a bull market is $> 1/2$.

Termination Linear Loops

Does this loop terminate?

Linear Loop

$x := 2^{-10}, y := 1$
until $\phi(x)$ do

$$\begin{bmatrix} x \\ y \end{bmatrix} := \begin{bmatrix} 2 & 0 \\ \frac{7}{4} & \frac{1}{4} \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

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Reachability problem

Given

- initial point: $x_0 \in \mathbb{Q}^d$,
- transition matrix: $A \in \mathbb{Q}^{d \times d}$,
- target set: $\mathcal{S} \subseteq \mathbb{R}^d$

decide if $\exists n \in \mathbb{N}.\ A^n x_0 \in \mathcal{S}$.

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Does this loop terminate?

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$x := 2^{-10}, y := 1$

until $x = 42$ and $y = 36$ do

$$\begin{bmatrix} x \\ y \end{bmatrix} := \begin{bmatrix} 2 & 0 \\ \frac{7}{4} & \frac{1}{4} \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

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- point:

$$\exists n \in \mathbb{N}\ A^n x_0 = y$$

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$$\exists n \in \mathbb{N}\ MA^n x_0 = b$$

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$$\exists n \in \mathbb{N}\ p(A^n x_0) \geqslant 0$$

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until $x^2 y \geqslant 1$ or $x = y$ do

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$$\exists n \in \mathbb{N}\ p(A^n x_0) \geqslant 0$$
- boolean combinations
- replace x_0 by an initial set $\mathcal{X}$
$$\exists x_0 \in \mathcal{X} \exists n \in \mathbb{N}\ A^n x_0 \in \mathcal{S}$$
$$\forall x_0 \in \mathcal{X} \exists n \in \mathbb{N}\ A^n x_0 \in \mathcal{S}$$

What is decidable about linear loops?

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Why do we need the dimension to be small?

From loops to recurrent sequences

Linear Loop

$x := x_0$
until $3x_1 - 7x_2 + 4x_3 = 0$ do
$x := Ax$

From loops to recurrent sequences

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$x := x_0$
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Half-space reachability

Given $x, y \in \mathbb{Q}^d$, $A \in \mathbb{Q}^{d \times d}$,
decide if $\exists n \in \mathbb{N}.\, y^T A^n x_0 = 0$.

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Consider the sequence $u_n = y^T A^n x$.

Lemma

There exists $a_0, \ldots, a_{d-1} \in \mathbb{Q}$ such that

$$u_{n+d} = a_{d-1} u_{n+d-1} + \cdots + a_0 u_n, \qquad \forall n \in \mathbb{N}.$$

In other words, $(u_n)_n$ is a linear recurrent sequence (LRS).

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- Fibonacci: $F_{n+2} = F_{n+1} + F_n$
- Pell numbers: $P_{n+2} = 2P_{n+1} + P_n$
- very common in combinatorics

From loops to recurrent sequences

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Lemma

For any LRS $(u_n)_n$, there exists x_0, y and A such that $u_n = y^T A^n x_0$.

Skolem and posivity problems

Linear recurrent sequence (LRS) of order d:

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Positivity Problem

Given a LRS $(u_n)_n$, decide if $u_n \geqslant 0$ for all $n \in \mathbb{N}$.

Harder than Skolem

Skolem-Mahler-Lech theorem

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Theorem (Skolem, Mahler, and Lech, 1933, 1953, 1957)

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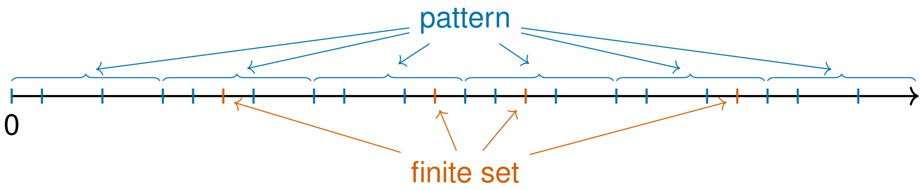

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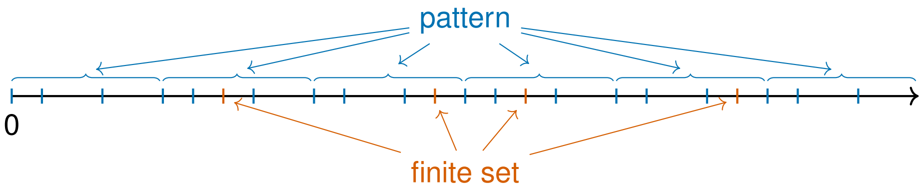

The regular patterm is computable. Nothing is known about the finite set: the proof is nonconstructive and uses p-adic analysis.

Skolem in low dimension

Theorem (Mignotte, Shorey, Tijdeman; Vereshchagin, 1985)

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How can we show hardness without proving undecidability?

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For any $x \in \mathbb{R}$, the (homogeneous Diophantine approximation) type

$$L(x) = \inf\left\{c \in \mathbb{R} : \left|x - \frac{n}{m}\right| < \frac{c}{m^2} \text{ for some } n, m \in \mathbb{Z}\right\}.$$

Intuitively, if $L(x) > 0$ then x is badly approximable by rationals.

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Intuitively, if $L(x) > 0$ then x is badly approximable by rationals. Almost nothing known for any concrete x except that $L(x) \in [0, 1/\sqrt{5}]$.

Theorem (Ouaknine and Worrell, 2013)

If Skolem is decidable at order 5 then one can approximate $L(x)$ *with arbitrary precision for a large class of numbers* x.

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Theorem (Ouaknine and Worrell, 2014)

The ultimate positivity problem is decidable for simple[‡] *LRS. It is at least as hard as deciding* $\exists\mathbb{R}$*.*

[‡] The associated characteristic polynomial has no repeated roots.

First-order queries on orbits

First-order orbit query (FOOQ): fully quantified first-order sentence whose atomic proposition are of the form

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Theorem (Almagor, Ouaknine and Worrell, 2021)

Given A and $\Phi(n)$ a FOOQ, it is decidable whether $\exists n \in \mathbb{N}.\, \Phi(n)$ in dimension $\leqslant 3$.

MSO model-checking

Given $x \in \mathbb{Q}^d$ and $A \in \mathbb{Q}^{n \times n}$ and $\mathcal{T}_1, \ldots, \mathcal{T}_k \subseteq \mathbb{R}^d$ semialgebraic sets.

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Let $\Sigma = \{0,1\}^k$ and define $w \in \Sigma^{\mathbb{N}}$ by

$$w_n = \left(A^n x \in \mathcal{T}_1, \ldots, A^n x \in \mathcal{T}_k\right).$$

Intuition: w_n records to which sets $A^n x$ belongs to at eact step n.

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Problem: given an MSO formula Ψ over $(\mathbb{N}, <)$, decide whether $w \models \Psi$.

Examples: $P_i(n)$ means $A^n x \in \mathcal{T}_i$

- $\mathcal{T}_i$ is reachable: $\exists n.\, P_i(n)$
- whenever $\mathcal{T}_i$ is visited $\mathcal{T}_j$ is visited some point later:

$$\forall n : P_i(n) \Rightarrow (\exists m > n : P_j(m))$$

MSO model-checking

Given $x \in \mathbb{Q}^d$ and $A \in \mathbb{Q}^{n \times n}$ and $\mathcal{T}_1, \dots, \mathcal{T}_k \subseteq \mathbb{R}^d$ semialgebraic sets.
Let $\Sigma = \{0,1\}^k$ and define $w \in \Sigma^{\mathbb{N}}$ by

$$w_n = \big(A^n x \in \mathcal{T}_1, \dots, A^n x \in \mathcal{T}_k\big).$$

Intuition: w_n records to which sets $A^n x$ belongs to at eact step n.
Problem: given an MSO formula Ψ over $(\mathbb{N}, <)$, decide whether $w \models \Psi$.

Examples: $P_i(n)$ means $A^n x \in \mathcal{T}_i$

- $\mathcal{T}_i$ is reachable: $\exists n.\, P_i(n)$
- whenever $\mathcal{T}_i$ is visited $\mathcal{T}_j$ is visited some point later:

$$\forall n : P_i(n) \Rightarrow (\exists m > n : P_j(m))$$

- in target $\mathcal{T}_i$ at every odd position:

$$\exists O \subseteq \mathbb{N} : \boxed{\text{formula to define odd numbers}} \wedge \forall x : x \in O \Rightarrow P_i(x)$$

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Theorem (Karimov, Lefaucheux, Ouaknine, Purser, Varonka, Whiteland, Worrell)

This is decidable if all $\mathcal{T}_i$ either have intrinsic dimension 1 or are included in a subspace of dimension 3.

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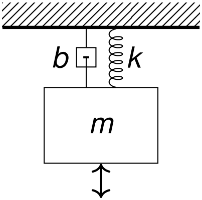
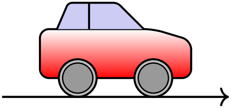
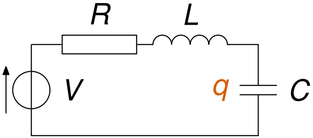
Continuous linear dynamical systems







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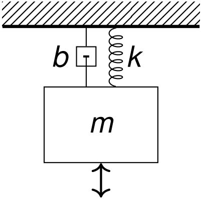

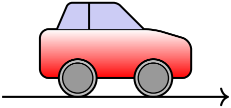

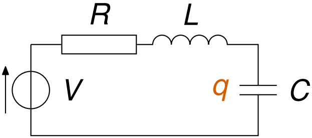

Linear differential equation:

$$x'(t) = Ax(t) \qquad x(0) = x_0$$

Example:

$$x'(t) = 7x(t)$$

$$\rightsquigarrow x(t) = e^{7t}$$

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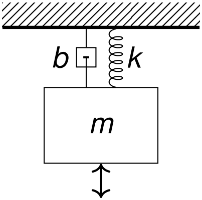
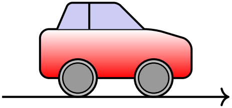
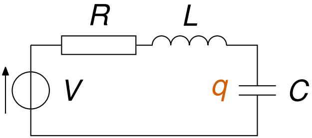






Linear differential equation:

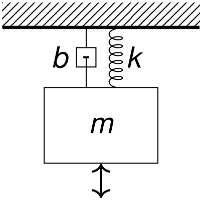
$$x'(t) = Ax(t) \qquad x(0) = x_0$$

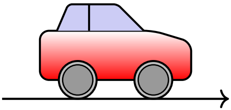
Example:

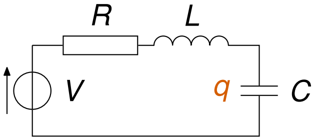
$$x'(t) = 7x(t) \qquad \begin{cases} x_1'(t) = x_2(t) \\ x_2'(t) = -x_1(t) \end{cases} \Leftrightarrow \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}' = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$\rightsquigarrow x(t) = e^{7t} \qquad \rightsquigarrow \begin{cases} x_1(t) = \sin(t) \\ x_2(t) = \cos(t) \end{cases}$$

Continuous linear dynamical systems







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General solution form:

$$x(t) = e^{At}x_0$$

$$\text{where } e^M = \sum_{n=0}^{\infty} \frac{M^n}{n!}$$

Continuous reachability

Continuous Skolem problem

Given x, y and A, decide if $\exists t \in \mathbb{R}$ such that $x^T e^{At} y = 0$.

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Given x, y and A, decide whether $x^T e^{At} y \geqslant 0$ for all $t \geqslant 0$.

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Continuous positivity is inter-reducible with continuous Skolem.

The decidability of all these problems is also open!

A link with number theory

Some reachability questions look like this :

$$\exists t \in \mathbb{R}.\ 42t^7 = 56 \ \wedge\ e^{3t} - e^t = 9$$

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Some reachability questions look like this (P, Q polynomials):

$$\exists t \in \mathbb{R}.\ P(t) = 0 \land Q(e^t) = 0$$

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If t is a nonzero algebraic number then e^t is transcendental.

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Theorem (Special case of Lindemann–Weierstrass)

If t is a nonzero algebraic number then e^t is transcendental.

- $P(t) = 0$ so t is algebraic (by definition)
- Lindemann–Weierstrass: e^t transcendental (unless $t = 0$)
- hence $Q(e^t) \neq 0$ (except maybe if $t = 0$)

Exponential polynomial

In general,

$$x^T e^{At} y = \sum_{i=1}^{d} P_i(t) e^{\lambda_i t}$$

where P_i polynomial, $\lambda_i \in \mathbb{C}$ eigenvalues of A.

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Theorem (Wilkie and MacIntyre)

If Schanuel's conjecture is true, then, for each $k \in \mathbb{N}$, the first-order theory of the structure $(\mathbb{R}, 0, 1, <, +, \cdot, \exp, \cos\restriction_{[0,k]}, \sin\restriction_{[0,k]})$ is decidable.

- algorithm always correct, only termination requires the conjecture

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What is Schanuel's conjecture?

Schanuel's conjecture

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If $z_1, \ldots, z_n$ that are linearly independent over $\mathbb{Q}$, then at least n numbers among $z_1, \ldots, z_n, e^{z_1}, \ldots, e^{z_n}$ are algebraically independent.

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Example: π and e are algebraically independent

$$z_1 = i\pi, z_2 = 1 \qquad \rightsquigarrow \qquad e^{z_1} = -1, e^{z_2} = e.$$

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Clearly z_1 and z_2 are linearly independent over $\mathbb{Q}$. So at least 2 of $i\pi, 1, -1, e$ are algebraically independent. But 1 is algebraic so π and e are algebraically independent.

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Summary:

- Schanuel implies that π, e, $\pi + e$, $e\pi$, ... are transcendental.
- π and e are known to be transcendental
- $\pi + e$ is **not known** to be transcendental

Continuous reachability

Bounded continuous Skolem problem: given x, y and A, decide if

- unbounded: $\exists t \in [0, 1]$ such that $x^T e^{At} y = 0$.
- bounded: $\exists t \in \mathbb{R}$ such that $x^T e^{At} y = 0$.

Theorem (Chonev, Ouaknine and Worrell, 2016)

The bounded continuous Skolem Problem is decidable subject to Schanuel's conjecture.

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The bounded continuous Skolem Problem is decidable subject to Schanuel's conjecture.

Theorem (Chonev, Ouaknine and Worrell, 2016)

If the (unbounded) continuous Skolem Problem is decidable then the Diophantine-approximation types of all real algebraic numbers is computable.

In other words: it requires new mathematics...

More complicated programs

Linear loop with if

$x := 2^{-10}$
$y := 1$
while $y \geqslant x$ do
  if $y \geqslant 2x$ then
$$\begin{bmatrix} x \\ y \end{bmatrix} := \begin{bmatrix} 2 & 0 \\ 1 & 4 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$
  else
$$\begin{bmatrix} x \\ y \end{bmatrix} := \begin{bmatrix} 2 & 3 \\ -3 & 7 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

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Reachability is trivially undecidable by simulating two counter automata

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$\rightsquigarrow$

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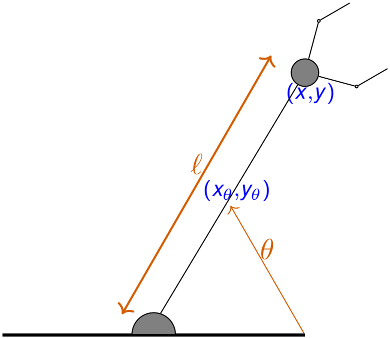
Reachability is trivially undecidable by simulating two counter automata

- Overapproximate behaviours
- Nondeterminic

Example: 2D robot



State: $\vec{u} = (x_\theta, y_\theta, x, y)$

Discretized actions:

- rotate arm by ψ
- change arm length by δ

$\rightsquigarrow$ Linear transformations

Rotate arm by ψ:

$$\begin{pmatrix} x \\ y \end{pmatrix} \leftarrow \begin{pmatrix} \cos\psi & -\sin\psi \\ \sin\psi & \cos\psi \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

$$\begin{pmatrix} x_\theta \\ y_\theta \end{pmatrix} \leftarrow \begin{pmatrix} \cos\psi & -\sin\psi \\ \sin\psi & \cos\psi \end{pmatrix} \begin{pmatrix} x_\theta \\ y_\theta \end{pmatrix}$$

Change arm length by δ:

$$\begin{pmatrix} x \\ y \end{pmatrix} \leftarrow \begin{pmatrix} x \\ y \end{pmatrix} + \delta \begin{pmatrix} x_\theta \\ y_\theta \end{pmatrix}$$

Matrix problems

Input: $A, C \in \mathbb{Q}^{d \times d}$ matrices
Output: $\exists n \in \mathbb{N}$ such that $A^n = C \quad ?$

Example: $\exists n \in \mathbb{N}$ such that

$$\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}^n = \begin{bmatrix} 1 & 100 \\ 0 & 1 \end{bmatrix} \quad ?$$

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Input: $A, C \in \mathbb{Q}^{d\times d}$ matrices
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Input: $A, B, C \in \mathbb{Q}^{d\times d}$ matrices
Output: $\exists n, m \in \mathbb{N}$ such that $A^n B^m = C$?

Example: $\exists n, m \in \mathbb{N}$ such that

$$\begin{bmatrix} 2 & 3 \\ 0 & 1 \end{bmatrix}^n \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ 0 & 1 \end{bmatrix}^m = \begin{bmatrix} 1 & 60 \\ 0 & 1 \end{bmatrix} \quad ?$$

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Input: $A_1, \ldots, A_k, C \in \mathbb{Q}^{d\times d}$ matrices
Output: $\exists n_1, \ldots, n_k \in \mathbb{N}$ such that $\prod_{i=1}^{k} A_i^{n_i} = C$?

Example: $\exists n, m, p \in \mathbb{N}$ such that

$$\begin{bmatrix} 2 & 3 \\ 0 & 1 \end{bmatrix}^n \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ 0 & 1 \end{bmatrix}^m \begin{bmatrix} 2 & 5 \\ 0 & 1 \end{bmatrix}^p = \begin{bmatrix} 81 & 260 \\ 0 & 1 \end{bmatrix} ?$$

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✓ Decidable if A_i commute × Undecidable in general

Input: $A_1, \ldots, A_k, C \in \mathbb{Q}^{d\times d}$ matrices
Output: $C \in \langle$semigroup generated by $A_1, \ldots, A_k\rangle$?

Semigroup: $\langle A_1, \ldots, A_k\rangle$ = all finite products of $A_1, \ldots, A_k$
Examples:

$$A_1A_3A_2 \quad A_1A_2A_1A_2 \quad A_3^8A_2A_1^3A_3^{42}$$

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Output: $\exists n, m \in \mathbb{N}$ such that $A^n B^m = C$? ✓ Decidable

Input: $A_1, \ldots, A_k, C \in \mathbb{Q}^{d\times d}$ matrices
Output: $\exists n_1, \ldots, n_k \in \mathbb{N}$ such that $\prod_{i=1}^{k} A_i^{n_i} = C$?
✓ Decidable if A_i commute × Undecidable in general

Input: $A_1, \ldots, A_k, C \in \mathbb{Q}^{d\times d}$ matrices
Output: $C \in \langle$semigroup generated by $A_1, \ldots, A_k\rangle$?
✓ Decidable if A_i commute × Undecidable in general

Semigroup: $\langle A_1, \ldots, A_k\rangle$ = all finite products of $A_1, \ldots, A_k$
Examples:

$$A_1A_3A_2 \quad A_1A_2A_1A_2 \quad A_3^8A_2A_1^3A_3^{42}$$

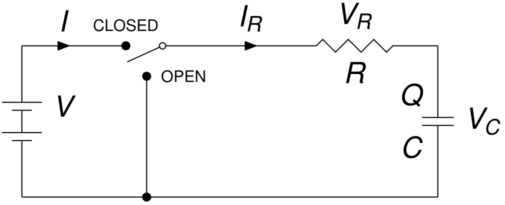
Discrete reachability problems

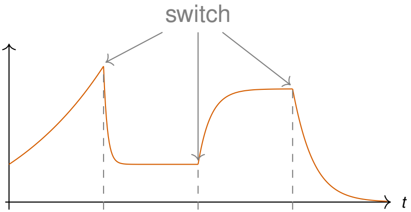
Every nontrivial extension of simple linear loops seems to lead to undecidable problems.

Discrete reachability problems

Every nontrivial extension of simple linear loops seems to lead to undecidable problems. What about the continuous setting?

RC circuit





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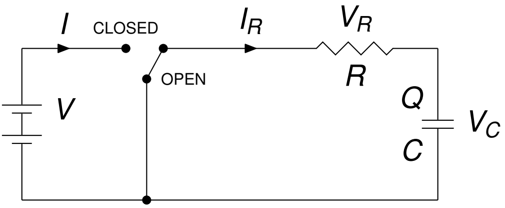
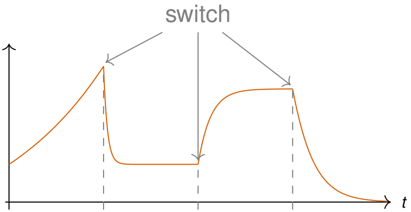
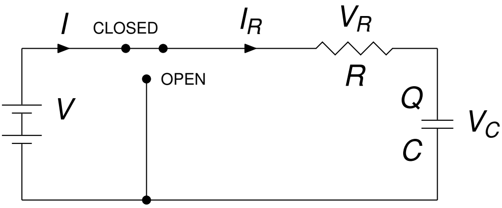


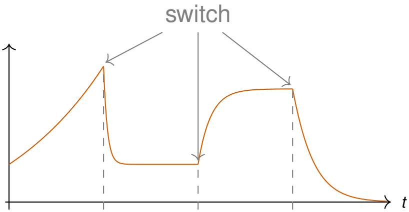


OPEN

$$\boxed{\begin{aligned} \dot{I} &= 0 \\ \dot{I}_R &= -\frac{1}{RC} I_R \\ \dot{V}_R &= -\frac{1}{C} I_R \\ \dot{Q} &= I_R \\ \dot{V}_C &= \frac{1}{C} I_R \end{aligned}}$$

RC circuit





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CLOSED

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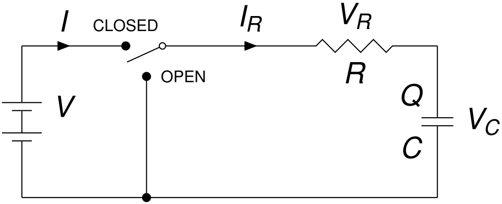
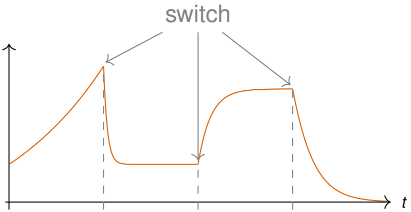




OPEN

$$\begin{aligned}\dot{I} &= 0\\ \dot{I}_R &= -\frac{1}{RC} I_R\\ \dot{V}_R &= -\frac{1}{C} I_R\\ \dot{Q} &= I_R\\ \dot{V}_C &= \frac{1}{C} I_R\end{aligned}$$

OPEN → CLOSED:

$$\begin{aligned}I &:= \frac{1}{R}(V - V_C)\\ I_R &:= \frac{1}{R}(V - V_C)\\ V_R &:= V - V_C\end{aligned}$$

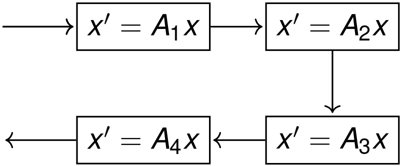
CLOSED → OPEN:

$$\begin{aligned}I &:= 0\\ I_R &:= -\frac{1}{R} V_C\\ V_R &:= -V_C\end{aligned}$$

CLOSED

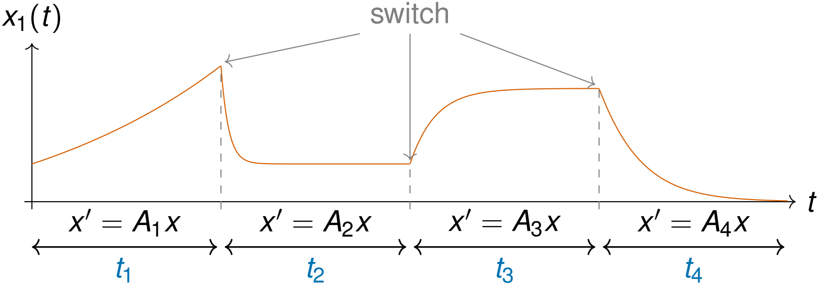
$$\begin{aligned}\dot{I} &= -\frac{1}{RC} I_R\\ \dot{I}_R &= -\frac{1}{RC} I_R\\ \dot{V}_R &= -\frac{1}{C} I_R\\ \dot{Q} &= I_R\\ \dot{V}_C &= \frac{1}{C} I_R\end{aligned}$$

Switching systems



Restricted hybrid system:

- linear dynamics
- no guards (nondeterministic)
- no discrete updates



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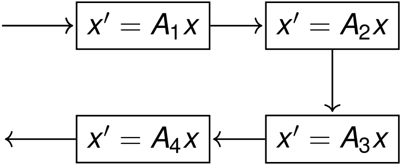

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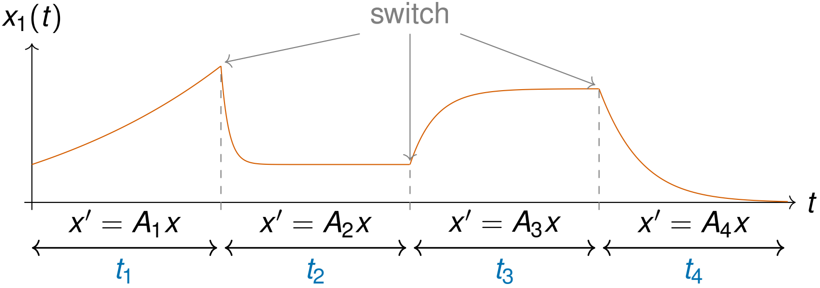
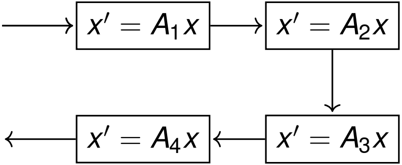


Dynamics:

$$e^{A_4 t_4} e^{A_3 t_3} e^{A_2 t_2} e^{A_1 t_1}$$



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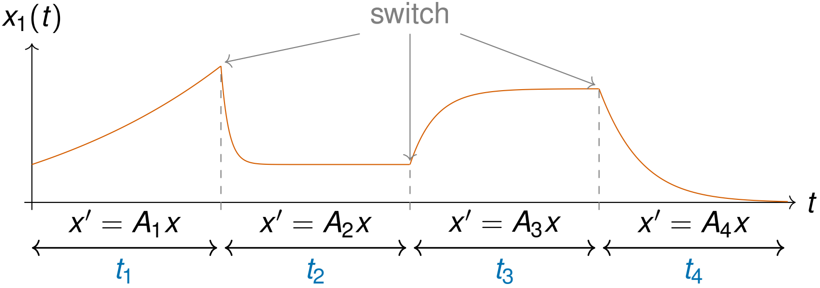

Problem:

$$e^{A_4 t_4} e^{A_3 t_3} e^{A_2 t_2} e^{A_1 t_1} = C \quad ?$$

What we control: $t_1, t_2, t_3, t_4 \in \mathbb{R}_{\geqslant 0}$

Related work in the continuous case

Input: $A, C \in \mathbb{Q}^{d\times d}$ matrices
Output: $\exists t \in \mathbb{R}$ such that $e^{At} = C$?

Example: $\exists t \in \mathbb{R}$ such that

$$\exp\left(\begin{bmatrix}1 & 1\\ 0 & 1\end{bmatrix} t\right) = \begin{bmatrix}1 & 100\\ 0 & 1\end{bmatrix} \quad ?$$

Related work in the continuous case

Input: $A, C \in \mathbb{Q}^{d\times d}$ matrices
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Input: $A, C \in \mathbb{Q}^{d\times d}$ matrices
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Input: $A, B, C \in \mathbb{Q}^{d\times d}$ matrices
Output: $\exists t, u \in \mathbb{N}$ such that $e^{At}e^{Bu} = C$?

Example: $\exists t, u \in \mathbb{R}$ such that

$$\exp\left(\begin{bmatrix} 2 & 3 \\ 0 & 1 \end{bmatrix} t\right) \exp\left(\begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ 0 & 1 \end{bmatrix} u\right) = \begin{bmatrix} 1 & 60 \\ 0 & 1 \end{bmatrix} \quad ?$$

Related work in the continuous case

Input: $A, C \in \mathbb{Q}^{d\times d}$ matrices
Output: $\exists t \in \mathbb{R}$ such that $e^{At} = C$? ✓ Decidable (PTIME)

Input: $A, B, C \in \mathbb{Q}^{d\times d}$ matrices
Output: $\exists t, u \in \mathbb{N}$ such that $e^{At}e^{Bu} = C$? × Unknown

Example: $\exists t, u \in \mathbb{R}$ such that

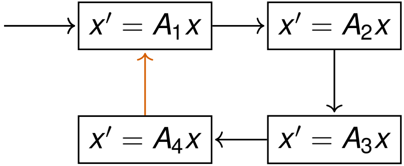
$$\exp\left(\begin{bmatrix} 2 & 3 \\ 0 & 1 \end{bmatrix} t\right) \exp\left(\begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ 0 & 1 \end{bmatrix} u\right) = \begin{bmatrix} 1 & 60 \\ 0 & 1 \end{bmatrix} \quad ?$$

Switching system

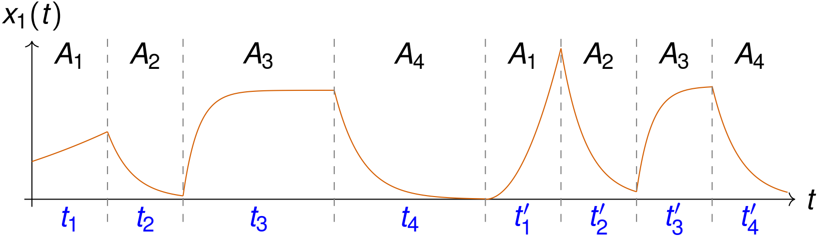
$$\begin{tikzcd}
\arrow[r] & \boxed{x' = A_1 x} \arrow[r] & \boxed{x' = A_2 x} \arrow[d] \\
& \boxed{x' = A_4 x} \arrow[u] & \boxed{x' = A_3 x} \arrow[l]
\end{tikzcd}$$

What about a loop ?

Switching system



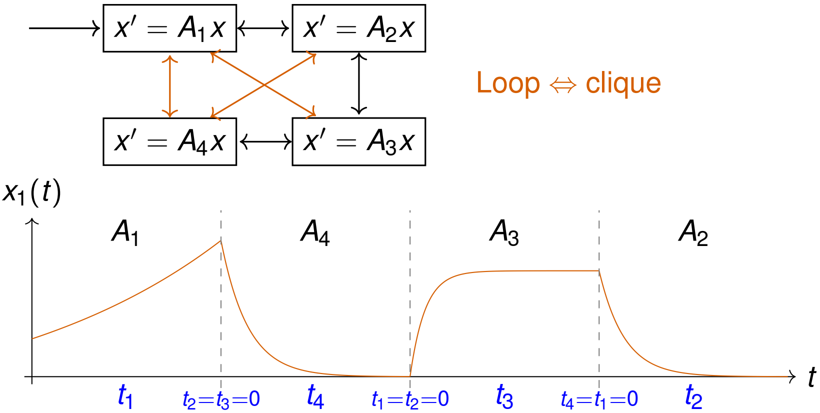
What about a loop ?



Dynamics:

$$e^{A_4 t'_4} e^{A_3 t'_3} e^{A_2 t'_2} e^{A_1 t'_1} e^{A_4 t_4} e^{A_3 t_3} e^{A_2 t_2} e^{A_1 t_1}$$

Switching system

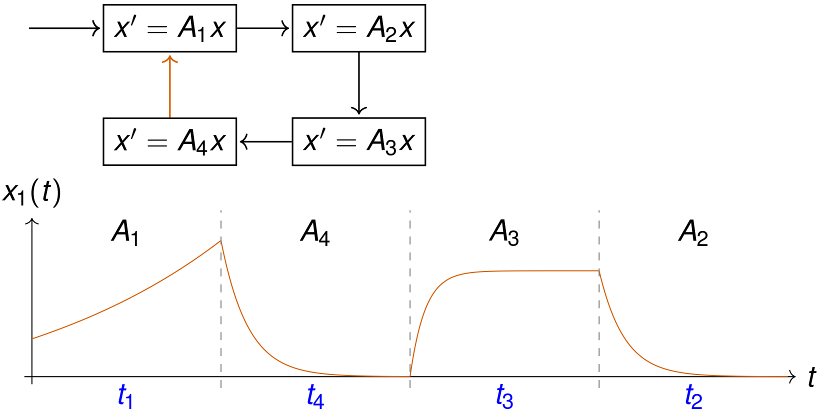


Loop $\Leftrightarrow$ clique

Remark:

zero time dynamics ($t_i = 0$) are allowed

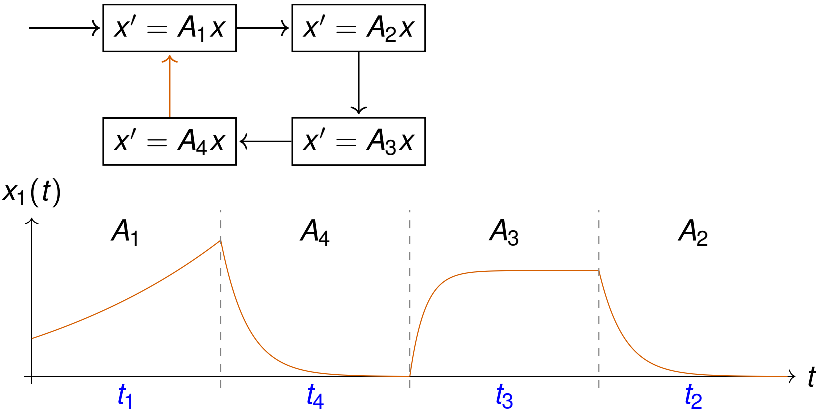
Switching system



Dynamics:

any finite product of $e^{A_i t}$ $\rightsquigarrow$ semigroup!

Switching system



Problem:

$$C \in \mathcal{G} \quad ?$$

where $\qquad \mathcal{G} = \langle \text{semigroup generated by } e^{A_i t} \text{ for all } t \geqslant 0 \rangle$

Reachability for switching systems

Input: $A_1, \dots, A_k, C \in \mathbb{Q}^{d \times d}$ matrices
Output: $\exists t_1, \dots, t_k \geqslant 0$ such that

$$\prod_{i=1}^{n} e^{A_i t_i} = C \quad ?$$

Input: $A_1, \dots, A_k, C \in \mathbb{Q}^{d \times d}$ matrices
Output:

$$C \in \langle \text{semigroup generated by } e^{A_1 t}, \dots, e^{A_k t} : t \geqslant 0 \rangle \quad ?$$

Theorem (Ouaknine, P, Sous-Pinto, Worrell)

Both problems are:

- *Undecidable in general*
- Decidable *when all the A_i commute*

Some words about the proof (commuting case)

Product Problem

$\exists t_1, \ldots, t_k \geqslant 0$ s.t. $\prod_{i=1}^{n} e^{A_i t_i} = C \quad ?$

equivalent

Semigroup Problem

$C \in \langle e^{A_1 t}, \ldots, e^{A_k t} : t \geqslant 0 \rangle \quad ?$

reduce

Integer Linear Programming

$\exists n \in \mathbb{Z}^d$ s.t. $\pi B n \leqslant s$

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Integer Linear Programming

$\exists n \in \mathbb{Z}^d$ s.t. $\pi B n \leqslant s$

⚠ s of the form:

$$a_0 + \log(a_1) + \cdots + \log(a_k)$$

✓ $B, a_0, \ldots, a_k$ are algebraic

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$$a_0 + \log(a_1) + \cdots + \log(a_k)$$

✓ $B, a_0, \ldots, a_k$ are algebraic

How did we get from reals to integers with π ?

$$e^{it} = \alpha \quad \Leftrightarrow \quad t \in \log(\alpha) + 2\pi\mathbb{Z}$$

Integer Linear Programming

$$\exists n \in \mathbb{Z}^d \text{ such that } \pi B n \leqslant s \quad ?$$

where s is a linear form in logarithms of algebraic numbers

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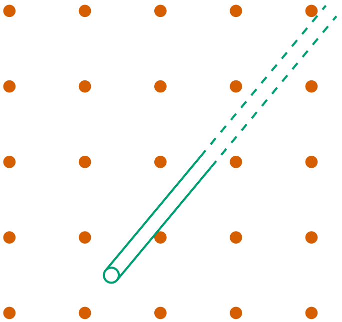
Key ingredient: Diophantine approximations

- Finding integer points in cones: Kronecker's theorem



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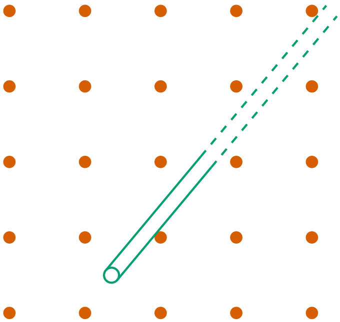


- Compare linear forms in logs: Baker's theorem

$$\sqrt{2} + \log\sqrt{3} - 3\log\sqrt{7} \quad \stackrel{?}{=} \quad 1 + \log 9 - \log \sqrt[42]{666}$$

Some words about the proof (general case)

Product Problem

$\exists t_1, \ldots, t_k \geqslant 0$ s.t. $\prod_{i=1}^{n} e^{A_i t_i} = C \quad ?$

reduce $\rightarrow$

Semigroup Problem

$C \in \langle e^{A_1 t}, \ldots, e^{A_k t} : t \geqslant 0 \rangle \quad ?$

reduce $\uparrow$

Hilbert's Tenth Problem

$\exists n \in \mathbb{Z}^d$ s.t. $p(n) = 0$

Theorem (Matiyasevich)

Hilbert's Tenth Problem is undecidable

Summary on reachability

Exact reachability is hard:

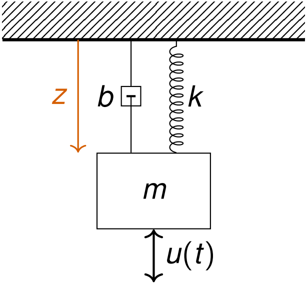
- Skolem/Positivity problem for linear loops (Open for 70 years)
- Every mild extension is undecidable
- Decidability requires very strong assumptions (commuting matrices)

Continuous vs discrete setting

- similar results
- different techniques
- continuous setting can leverage powerful results/conjectures

Control Theory

Example: mass-spring-damper system



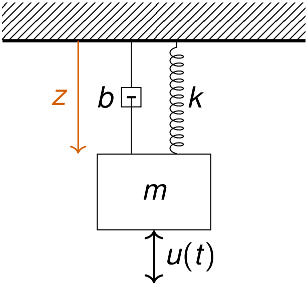
State: $X = z \in \mathbb{R}$

Equation of motion:

$$mz'' = -kz - bz' + mg + u$$

Model with external input $u(t)$

Example: mass-spring-damper system



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$\rightarrow$ Affine but not first order

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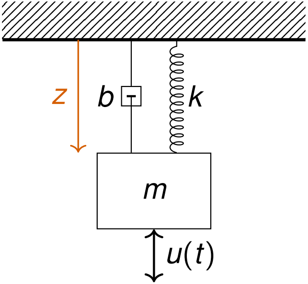

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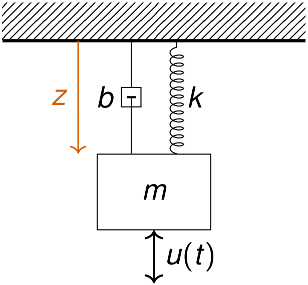
$\rightarrow$ Affine but not first order

State: $X = (z, z', 1) \in \mathbb{R}^3$

Equation of motion:

$$\begin{bmatrix} z \\ z' \\ 1 \end{bmatrix}' = \begin{bmatrix} z' \\ -\frac{k}{m}z - \frac{b}{m}z' + g + \frac{1}{m}u \\ 0 \end{bmatrix}$$

Example: mass-spring-damper system



Model with external input $u(t)$
$\rightarrow$ Linear time invariant system

$$X' = AX + Bu$$

with some constraints on u.

State: $X = z \in \mathbb{R}$

Equation of motion:

$$mz'' = -kz - bz' + mg + u$$

$\rightarrow$ Affine but not first order

State: $X = (z, z', 1) \in \mathbb{R}^3$

Equation of motion:

$$\begin{bmatrix} z \\ z' \\ 1 \end{bmatrix}' = \begin{bmatrix} z' \\ -\frac{k}{m}z - \frac{b}{m}z' + g + \frac{1}{m}u \\ 0 \end{bmatrix}$$

A very simple example

A simplified one-dimensional car: control acceleration $u(t)$

$$x''(t) = u(t)$$

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Starting at $x(0) = a$, want to reach and stop at $x = b$:

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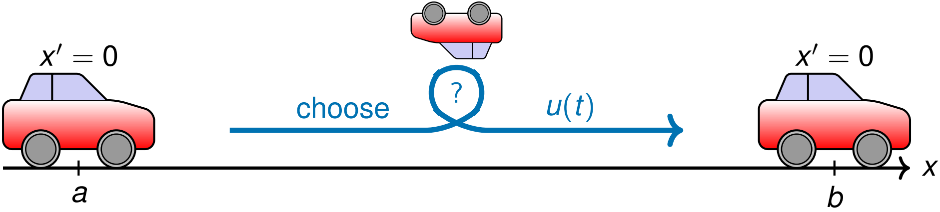

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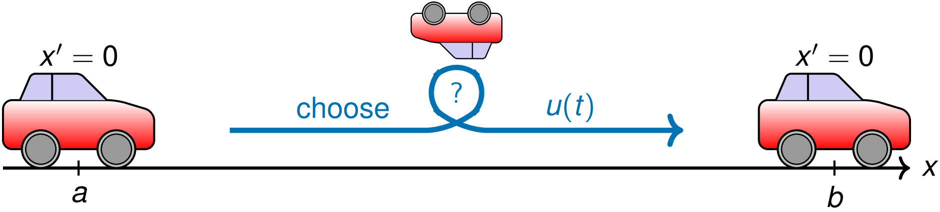
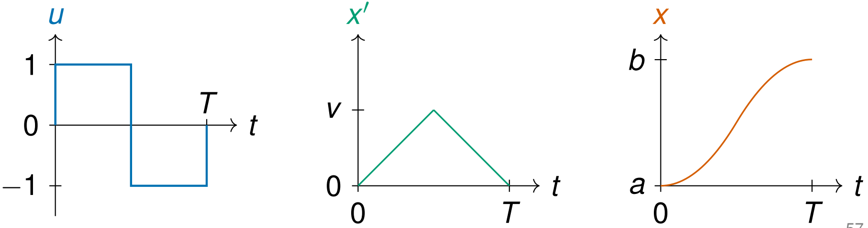


Possible solution:



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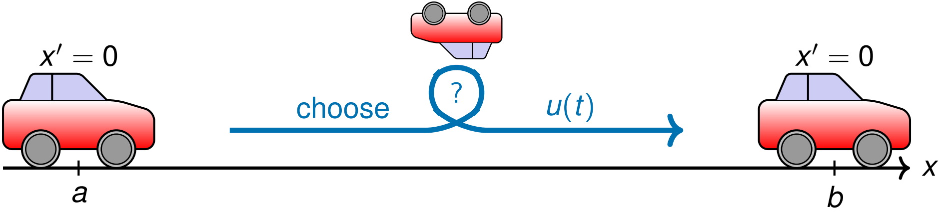
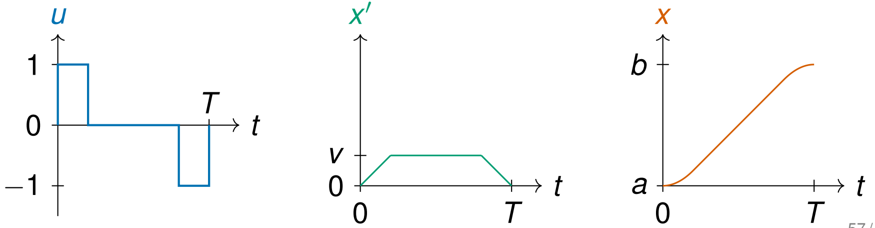


More realistic solution:



A very simple example

A simplified one-dimensional car: control acceleration $u(t)$

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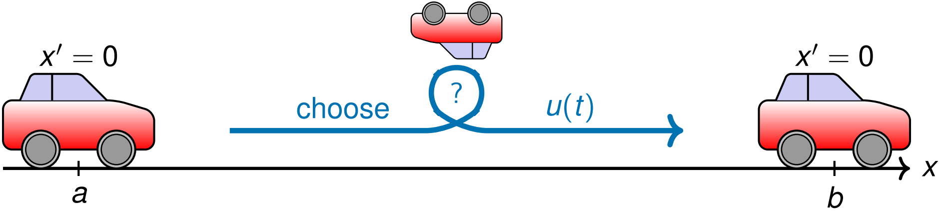

Rephrasing the problem:

$$\begin{cases} x' = y \\ y' = u \end{cases} \Leftrightarrow \begin{bmatrix} x \\ y \end{bmatrix}' = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} 0 \\ u \end{bmatrix} \Leftrightarrow X' = AX + U$$

Starting from $(x, y) = (a, 0)$, try to reach $(x, y) = (b, 0)$.

This is a point-to-point reachability problem.

The problem

LTI Reachability problem

- a source $y \in \mathbb{Q}^n$,
- a target $z \in \mathbb{Q}^n$,
- a transition matrix $A \in \mathbb{Q}^{n \times n}$,
- a set of controls $U \subseteq \mathbb{R}^n$,

decide if $\exists T \geqslant 0$, $u : [0, T] \to U$ *measurable* such that $x(T) = z$ where

$$x(0) = y, \qquad x'(t) = Ax(t) + u(t) \quad \text{for } t \in [0, T].$$

The problem

LTI Reachability problem

- a source $y \in \mathbb{Q}^n$,
- a target $z \in \mathbb{Q}^n$,
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$$x(0) = y, \qquad x'(t) = Ax(t) + u(t) \quad \text{for } t \in [0, T].$$

Warning: u **does not need to be "describable"**, *e.g.* piecewise polynomial. Otherwise, completely changes the nature of the problem.

Bigger picture

Continuous Reachability problem

- a source $y \in \mathbb{Q}^n$,
- a target $z \in \mathbb{Q}^n$,
- a transition function f,
- a set of controls $U \subseteq \mathbb{R}^m$,

decide if $\exists T \geqslant 0$, $u : [0, T] \to U$ *measurable* such that $x(T) = z$ where

$$x(0) = y, \qquad x'(t) = f(t, x(t), u(t)) \quad \text{for } t \in [0, T].$$

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But do they really?

Existing work

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But also:

- assumptions on A (typically spectral)
- assumptions on U
- restrictions on acceptable u

Two known extreme cases

- When we have no control:

$$U = \{0\} \quad \text{and} \quad x'(t) = Ax + u(t) \qquad \Leftrightarrow \qquad x(t) = e^{At}x(0).$$

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Theorem (Folklore)

Given $y, z \in \mathbb{Q}^n$ and $A \in \mathbb{Q}^{n\times n}$, $B \in \mathbb{Q}^{n\times m}$, it is decidable whether $\exists T \geqslant 0$ and $u : [0, T] \to B\mathbb{R}^m$ measurable such that $x(0) = y$ and $x(T) = z$ where

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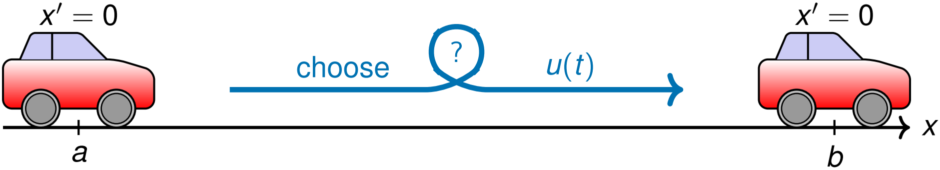
Back to the future

A simplified one-dimensional car: control acceleration $u(t)$

$$x''(t) = u(t)$$

Starting at $x(0) = a$, want to reach and stop at $x = b$:



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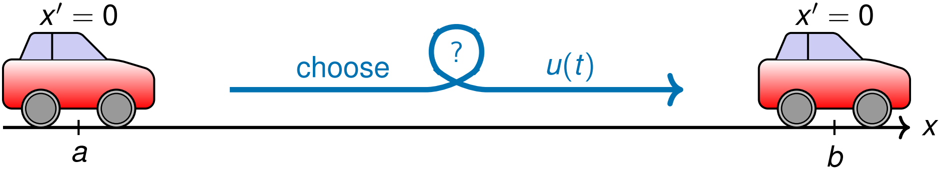

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Very few decidability results in the literature in this case.

Our results: decidability

LTI Zonotope Null-Reachability problem

Given a matrix $A \in \mathbb{Q}^{n\times n}$, a set of controls $U = B[-1,1]^m$, a target $z \in \mathbb{Q}^n$, decide if $\exists T \geqslant 0$, $u : [0,T] \to U$ such that $x(T) = z$ where

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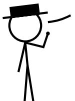

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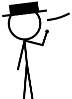

Well, that was underwhelming...

Are you sure you cannot do better?

Our results: conditional decidability

Schanuel's conjecture

A deep conjecture in transcendental number theory. Widely believed to be true and totally open.

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Theorem (Wilkie and MacIntyre)

If Schanuel's conjecture is true, then, for each $k \in \mathbb{N}$, the first-order theory of the structure $(\mathbb{R}, 0, 1, <, +, \cdot, \exp, \cos\restriction_{[0,k]}, \sin\restriction_{[0,k]})$ is decidable.

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Study generalization:

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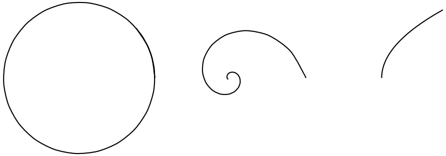
This is a well-known “hard” problem.

Hardness (cont.)

Taking $U = \{0\}$ is cheating:

- when $U = \{0\}$, reachable set is closed (or closed minus a point)



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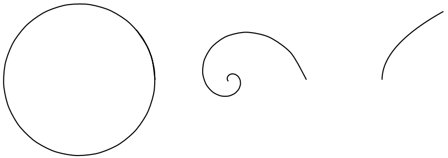
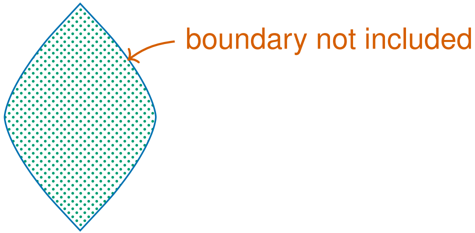


- when $U = B[-1, 1]^m$, reachable set is open



This is completely different!

Our results: hardness

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Continuous Nontangential Skolem problem

Given a matrix $A \in \mathbb{Q}^{n\times n}$ and $c, x_0 \in \mathbb{Q}^n$, decide if $\exists T \geqslant 0$ such that $f(t) = 0$ and $f'(t) \neq 0$ where $f(t) = c^T e^{At} x_0 = 0$.

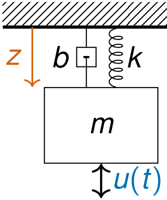
It is essentially as hard as the Continuous Skolem problem.

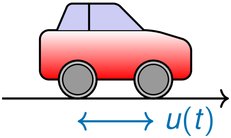
Conclusion (continuous case)

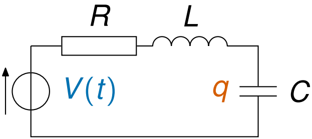
LTI reachability problem: find T and u such that

$$x(0) = 0, \qquad x'(t) = Ax(t) + Bu(t), \qquad u(t) \in [-1, 1]^m$$

satisfies $x(T) = $ target. Very natural problem in control theory.







Point reachability is

- decidable in dimension 2 or with spectral constraints,
- conditionally decidable with real eigenvalues,
- conditionally decidable in bounded time,

Set reachability is Nontangential Continuous Skolem hard.

The continuous case is much harder than expected. What about the discrete case?

The problem

LTI-REACHABILITY

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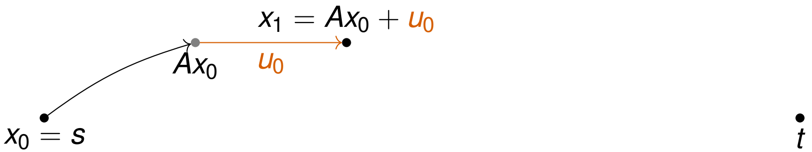

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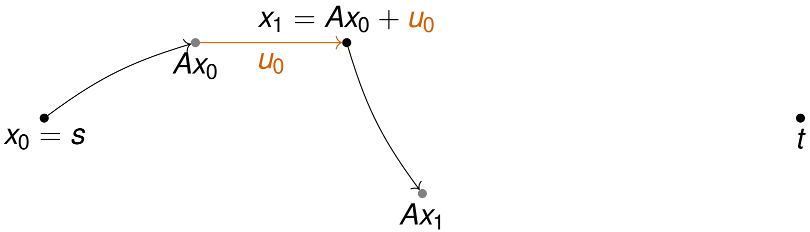

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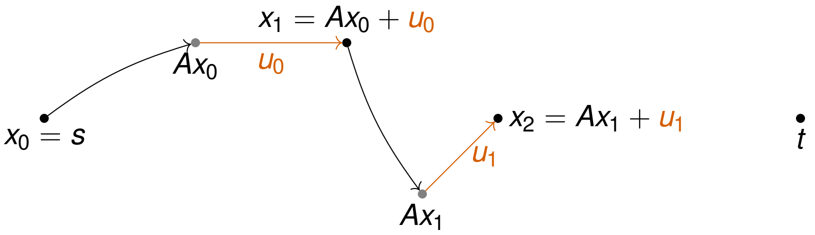

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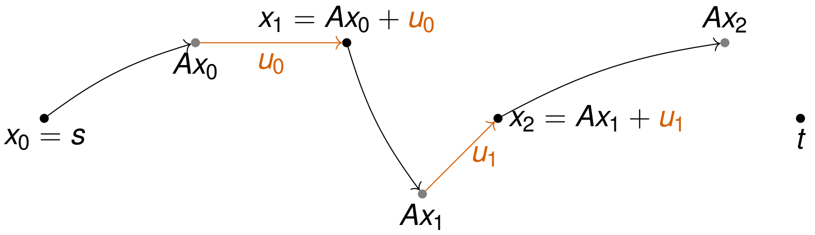

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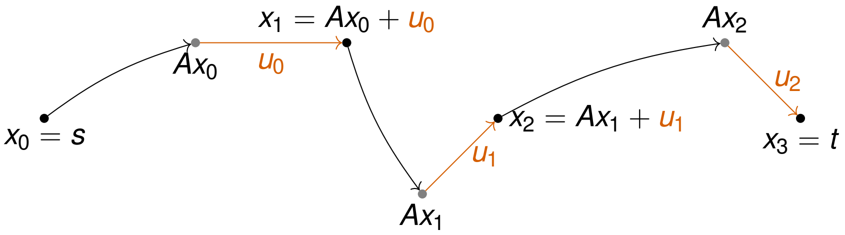

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Almost no exact results for other classes of U in particular when U is bounded (which is the most natural case).

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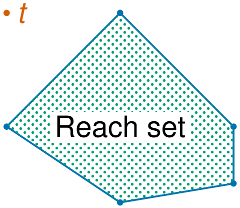

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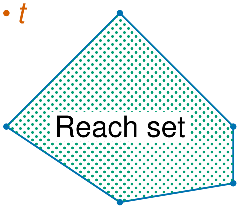

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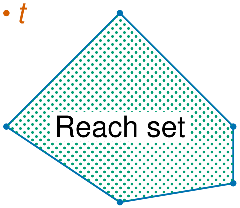

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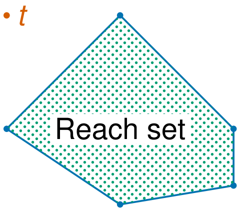

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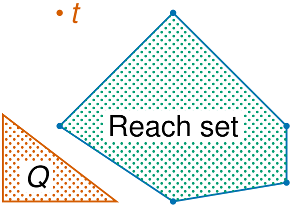
Remark: in fact we can decide reachability to a convex polytope Q.



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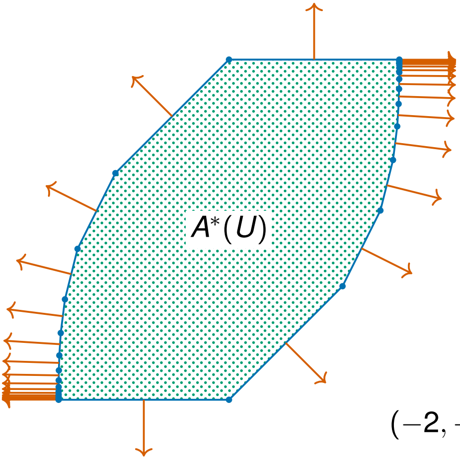
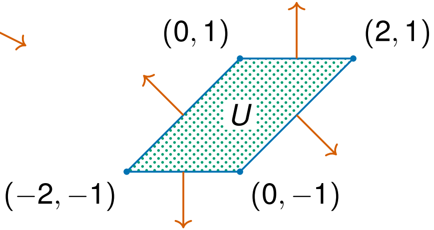
Why is this problem hard

The reachable set $A^*(U)$ can have **infinitely** many faces.



$$A = \begin{bmatrix} \frac{1}{3} & 0 \\ 0 & \frac{2}{3} \end{bmatrix}$$



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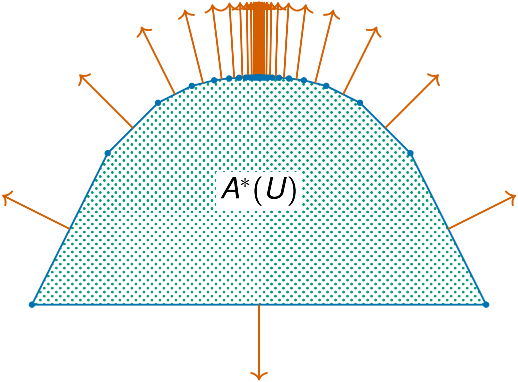
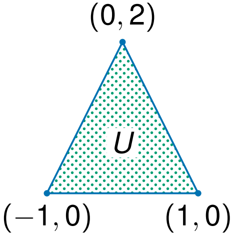
The reachable set $A^*(U)$ can have **faces of lower dimension**: the "top" extreme point does not belong to any facet.



$$A = \begin{bmatrix} \frac{2}{3} & 0 \\ 0 & \frac{1}{3} \end{bmatrix}$$



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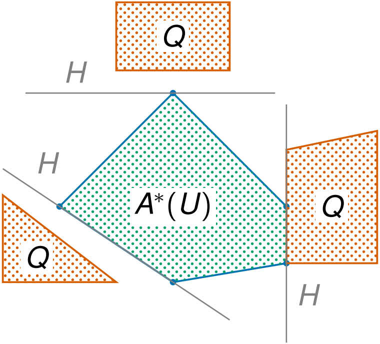

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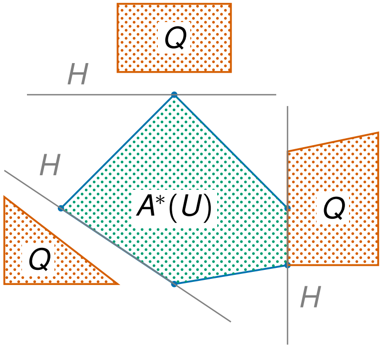

Further difficulty: a separating hyperplane may not be supported by a facet of either $A^*(U)$ or Q.

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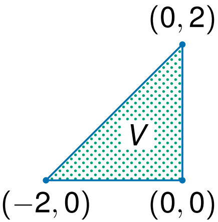
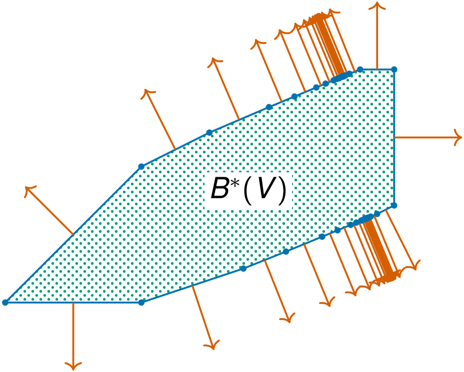
$$B = \begin{bmatrix} \frac{2}{3} & \frac{1}{3} \\ \frac{1}{3} & 0 \end{bmatrix}$$





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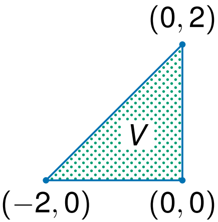

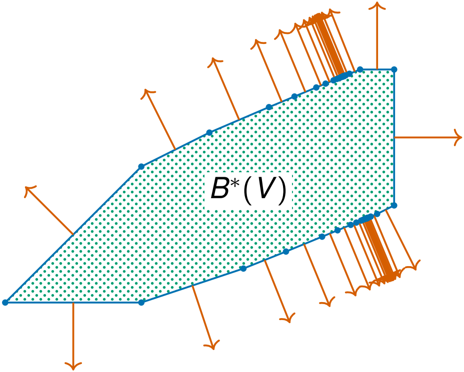

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Theorem (Non-reachable instances)

There is a separating hyperplane with algebraic coefficients.

Conclusion on control

Exact reachability for LTI systems:

- decidability crucially depends on the shape of the control set
- even with convex bounded inputs, the problem is very hard (Skolem/Positivity, open for 70 years)
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Despite an extensive literature in control theory, the decidability control problems is still very open.

Invariant Synthesis

Does this program halt?

Affine program

$x := 2^{-10}$
$y := 1$
while $y \geqslant x$ do

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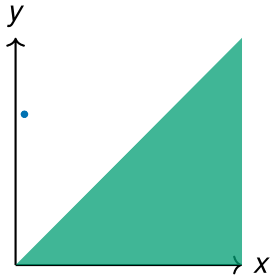

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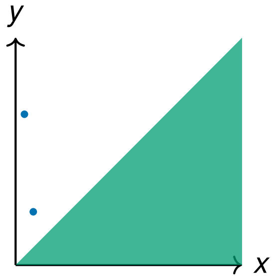

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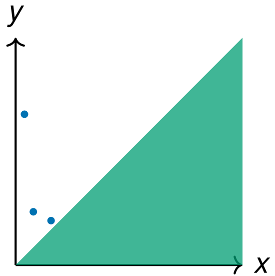

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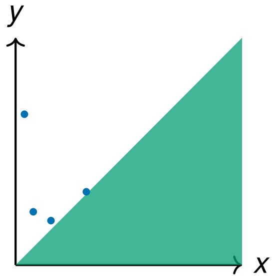

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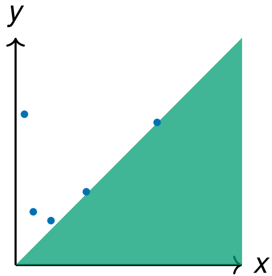

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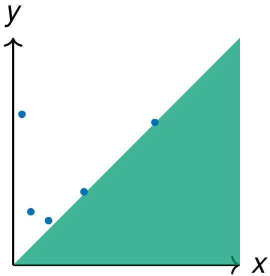
$$x^2 y - x^3 = \frac{1023}{1073741824} \tag{2}$$



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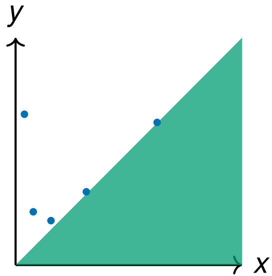

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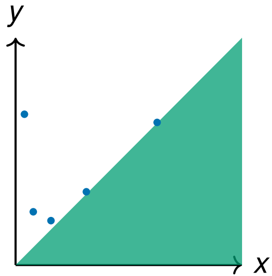

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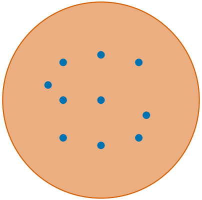
Invariants

invariant = overapproximation of the reachable states



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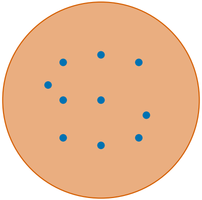
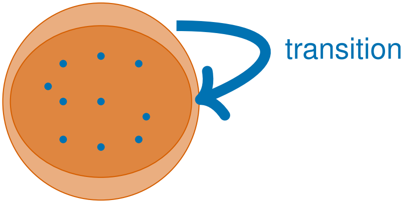


inductive invariant = invariant preserved by the transition relation



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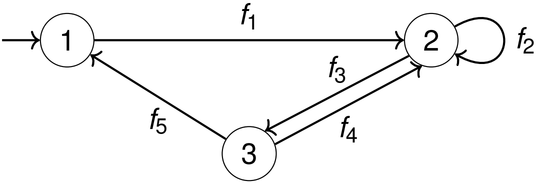

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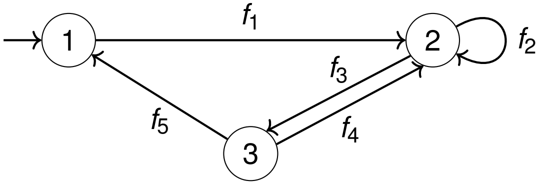
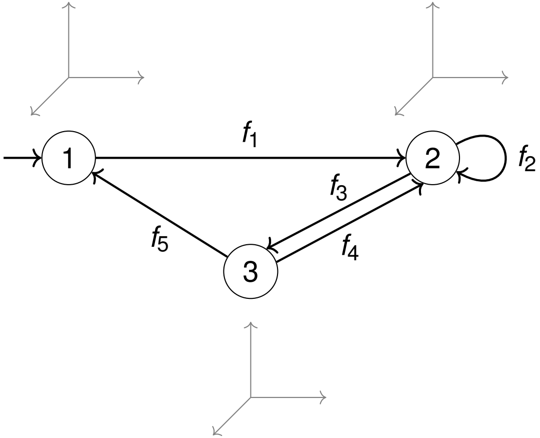
x, y, z range over $\mathbb{Q}$ $\qquad\qquad$ $f_i : \mathbb{R}^3 \to \mathbb{R}^3$



Inductive invariants: example

x, y, z range over $\mathbb{Q}$ $\qquad f_i : \mathbb{R}^3 \to \mathbb{R}^3$



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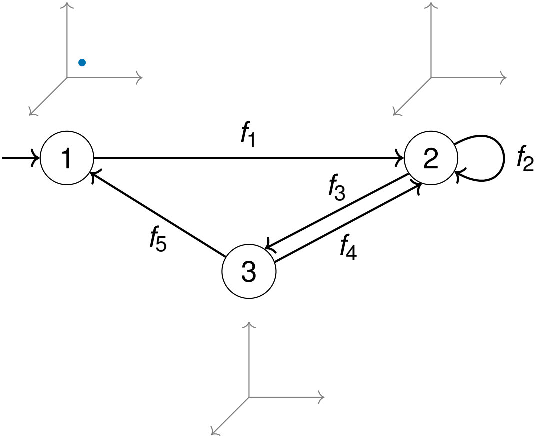

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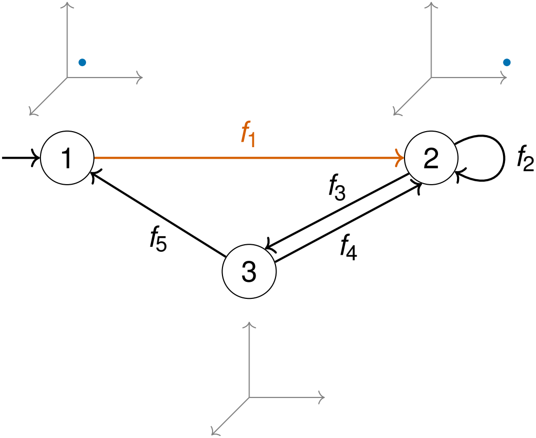

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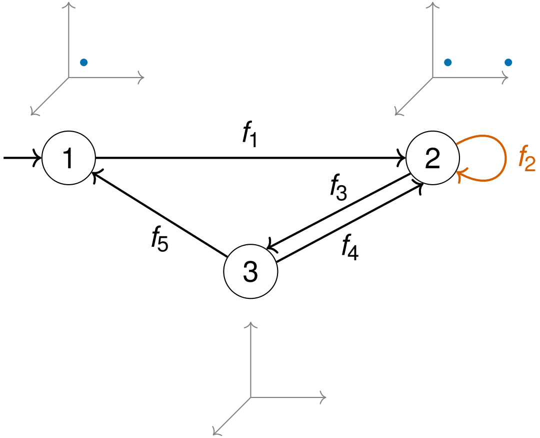

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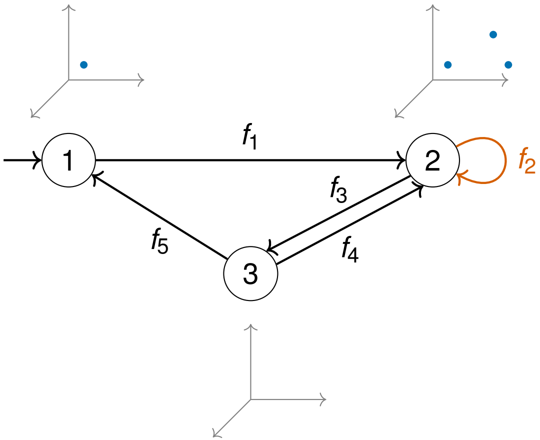

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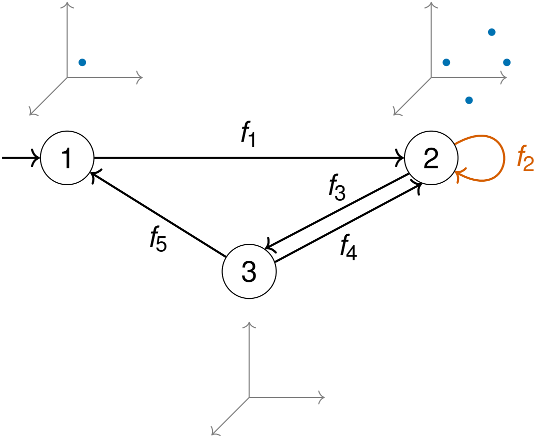

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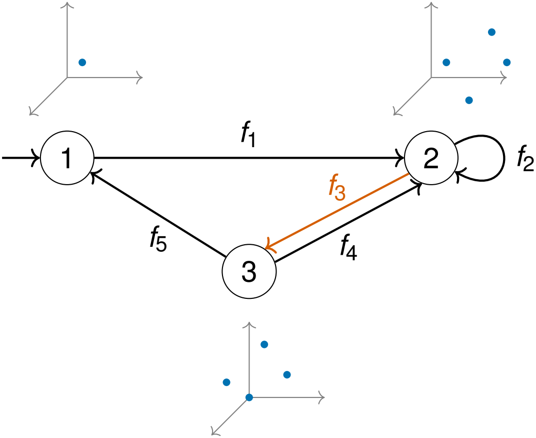

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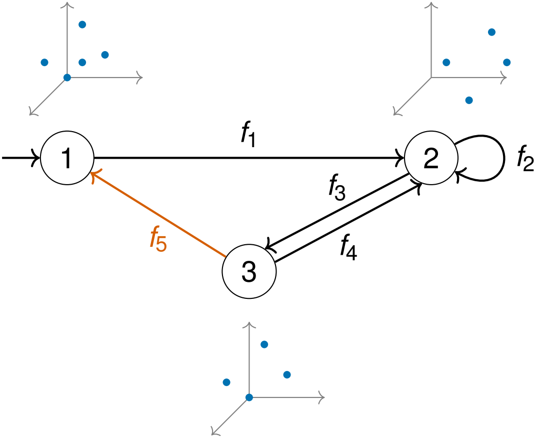

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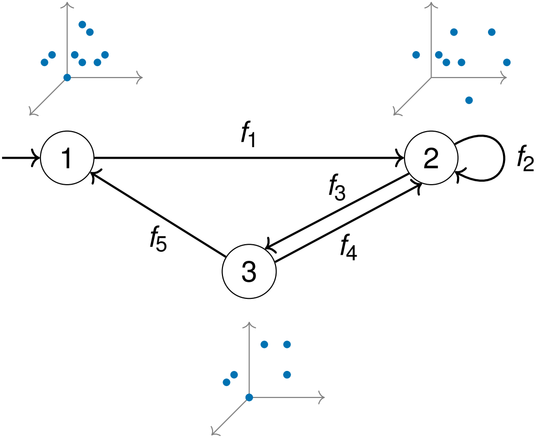

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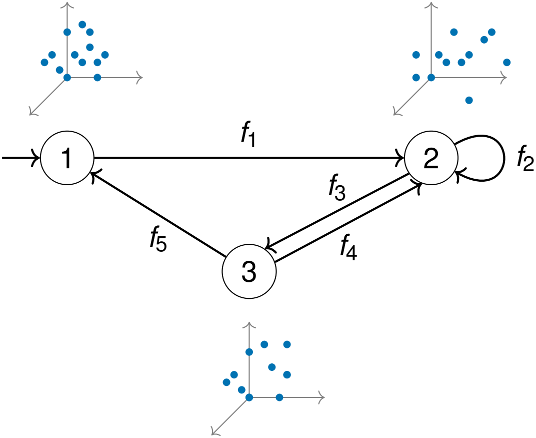

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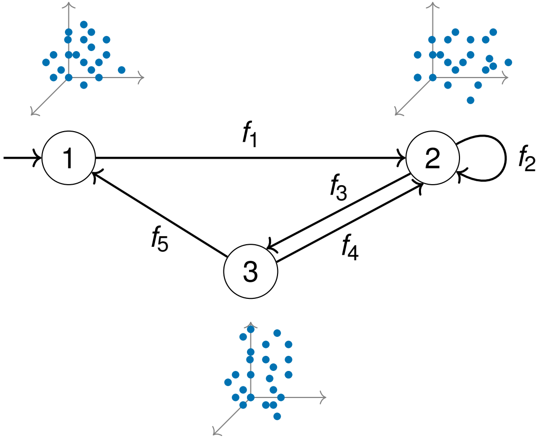

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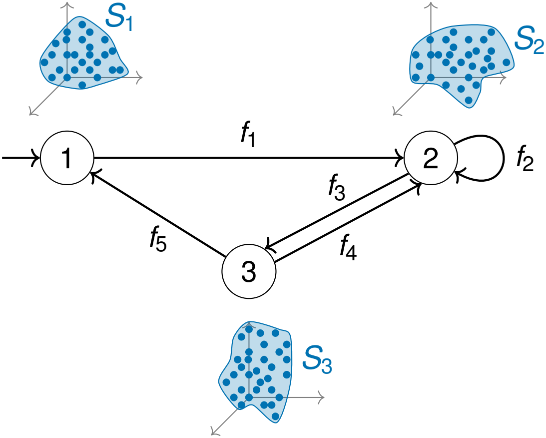
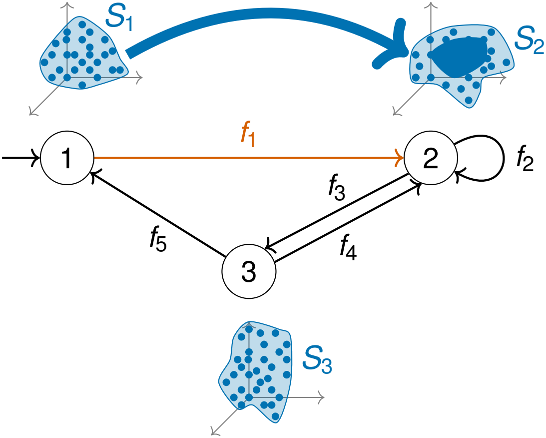


S_1, S_2, S_3 are the reachable states

Inductive invariants: example

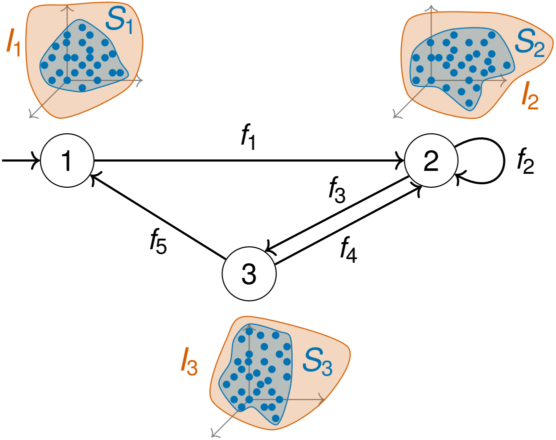
x, y, z range over $\mathbb{Q}$ $\qquad f_i : \mathbb{R}^3 \to \mathbb{R}^3$



S_1, S_2, S_3 is also an inductive invariant

Inductive invariants: example

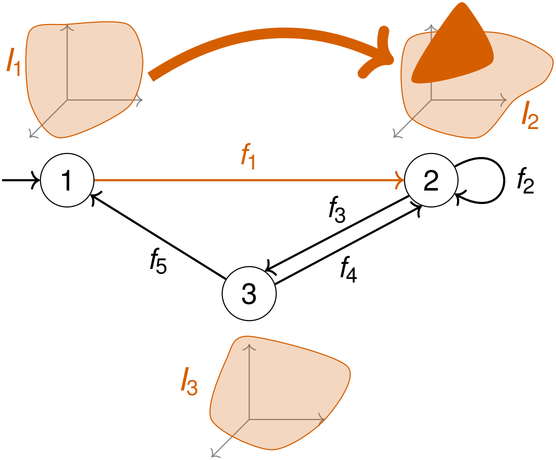
x, y, z range over $\mathbb{Q}$ $f_i : \mathbb{R}^3 \to \mathbb{R}^3$



I_1, I_2, I_3 is an invariant

Inductive invariants: example

x, y, z range over $\mathbb{Q}$ $f_i : \mathbb{R}^3 \to \mathbb{R}^3$



I_1, I_2, I_3 is **NOT** an inductive invariant

Inductive invariants: example

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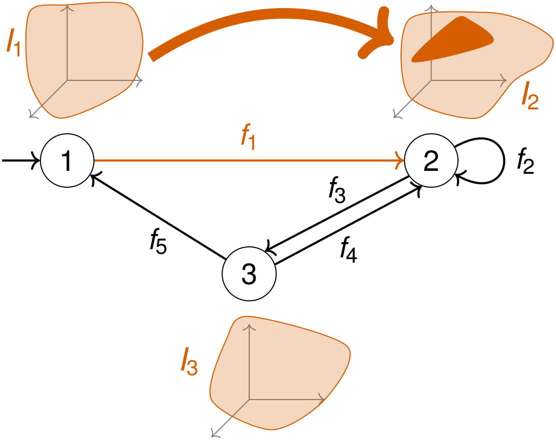
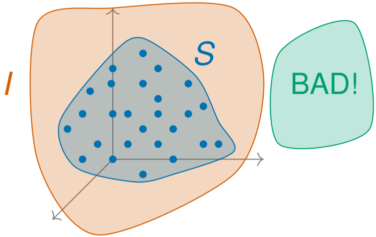


I_1, I_2, I_3 is an inductive invariant

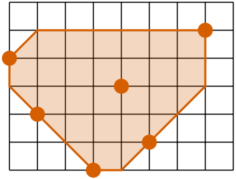
Why Invariants?



The classical approach to the verification of temporal safety properties of programs requires the construction of ***inductive invariants*** *[...].* ***Automation of this construction is the main challenge in program verification.***

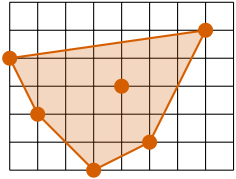
D. Beyer, T. Henzinger, R. Majumdar, and A. Rybalchenko
Invariant Synthesis for Combined Theories, 2007

Which invariants?



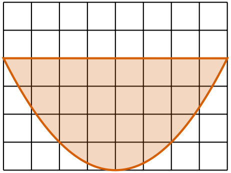
Octagons

$\leqslant$



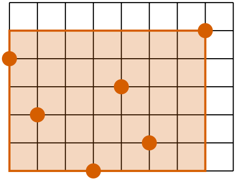
Polyhedrons

$\leqslant$

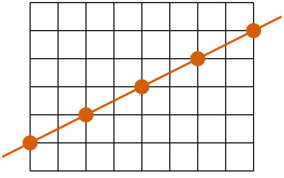


Semialgebraic sets

$\rotatebox{90}{$\leqslant$}$ $\rotatebox{90}{$\leqslant$}$ $\rotatebox{90}{$\leqslant$}$

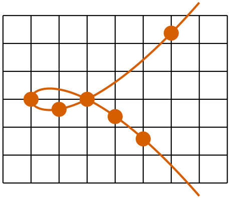


Intervals



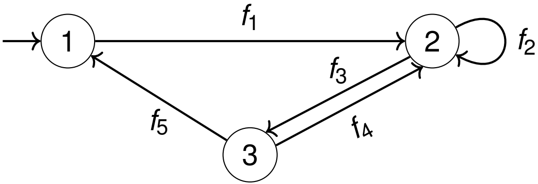
Affine/linear sets

$\leqslant$



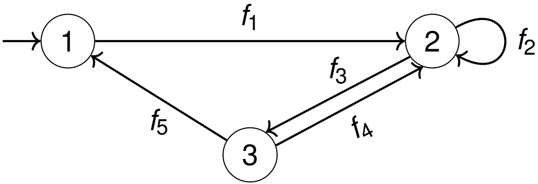
Algebraic sets =
polynomial equalities

Affine programs



Affine programs

- Nondeterministic branching (no guards)



Affine programs

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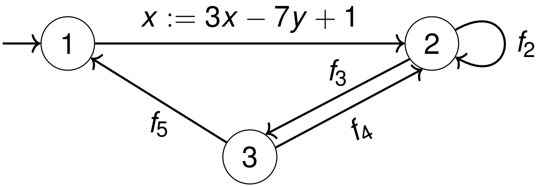
- All assignments are affine



Affine programs

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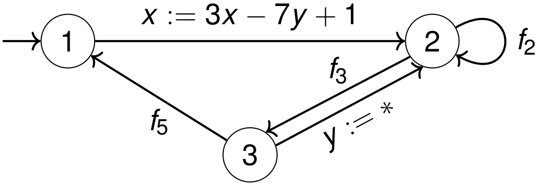
- Allow nondeterministic assignments ($x := *$)



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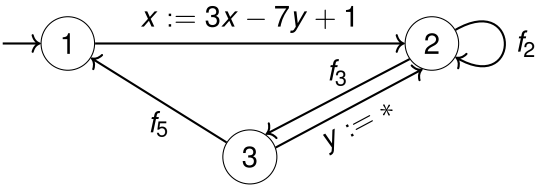

- Can overapproximate complex programs

Affine programs

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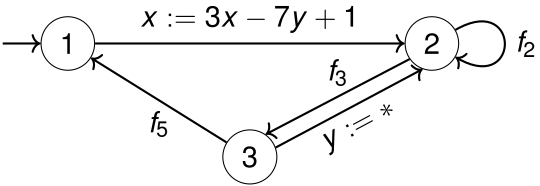


- Can overapproximate complex programs
- Covers existing formalisms:
  probabilistic, quantum, quantitative automata

Karr's Algorithm

Affine Relationships Among Variables of a Program*

Michael Karr

Received May 8, 1974

Summary. Several optimizations of programs can be performed when in certain regions of a program equality relationships hold between a linear combination of the variables of the program and a constant. This paper presents a practical approach to detecting these relationships by considering the problem from the viewpoint of linear algebra. Key to the practicality of this approach is an algorithm for the calculation of the "sum" of linear subspaces.

Theorem (Karr 76)

There is an algorithm which computes, for any given affine program over $\mathbb{Q}$, its strongest affine inductive invariant.

Discovering Affine Equalities Using Random Interpretation

Sumit Gulwani George C. Necula
University of California, Berkeley
{gulwani,necula}@cs.berkeley.edu

ABSTRACT

We present a new polynomial-time randomized algorithm for discovering affine equalities involving variables in a program.

Keywords

Affine Relationships, Linear Equalities, Random Interpretation, Randomized Algorithm

Some polynomial invariants

A Note on Karr's Algorithm

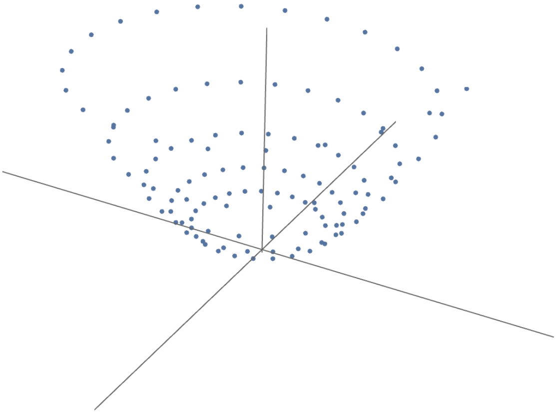
Markus Müller-Olm[1⋆] and Helmut Seidl[2]

Abstract. We give a simple formulation of Karr's algorithm for computing all affine relationships in affine programs. This simplified algorithm runs in time $\mathcal{O}(nk^3)$ where n is the program size and k is the number of program variables assuming unit cost for arithmetic operations. This improves upon the original formulation by a factor of k. Moreover, our re-formulation avoids exponential growth of the lengths of intermediately occurring numbers (in binary representation) and uses less complicated elementary operations. We also describe a generalization that determines all polynomial relations up to degree d in time $\mathcal{O}(nk^{3d})$.

Theorem (ICALP 2004)

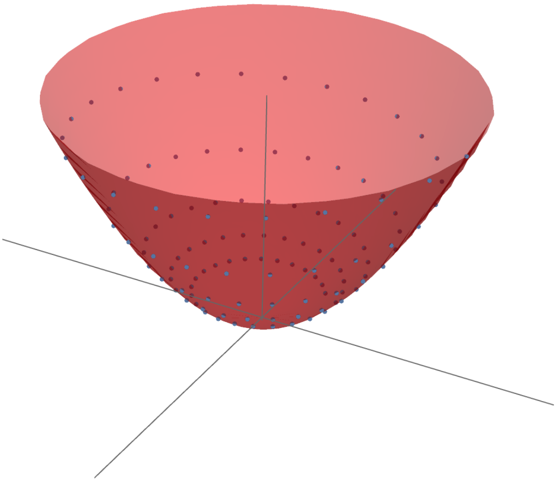
There is an algorithm which computes, for any given affine program over $\mathbb{Q}$, all its polynomial inductive invariants up to any ***fixed degree*** *d.*

Why fixed degree is not enough



Why fixed degree is not enough

- Paraboloid $z = x^2 + y^2$



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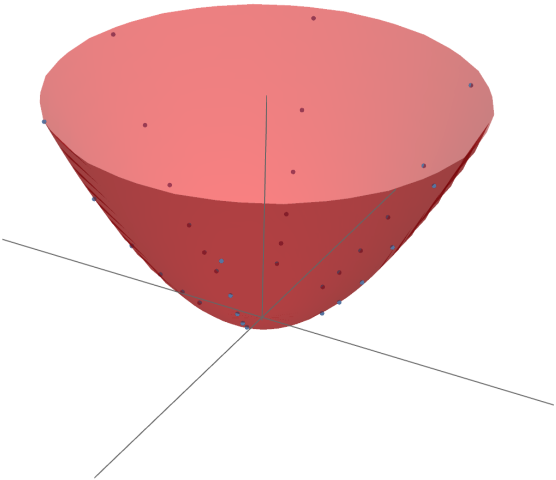
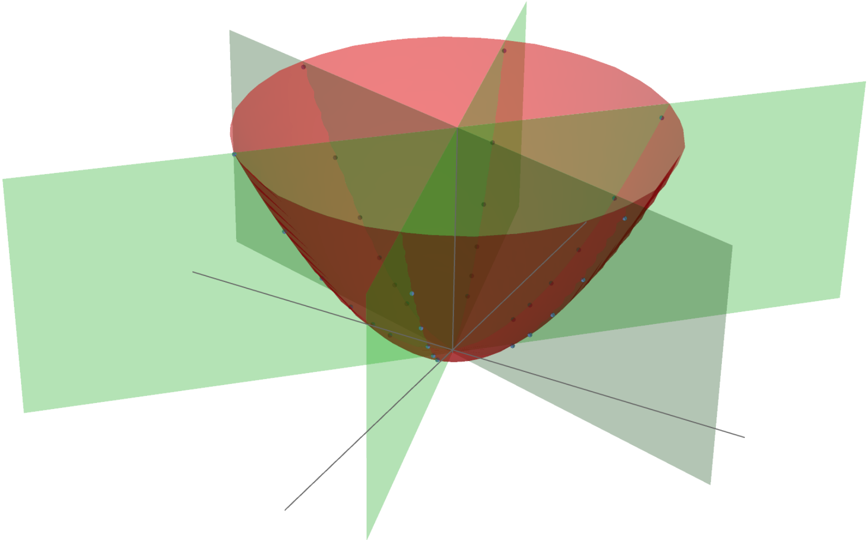


Why fixed degree is not enough

- Paraboloid $\quad z = x^2 + y^2$
- Union of 3 hyperplanes $\quad (x - y)(10y + x)(y + 10x) = 0$



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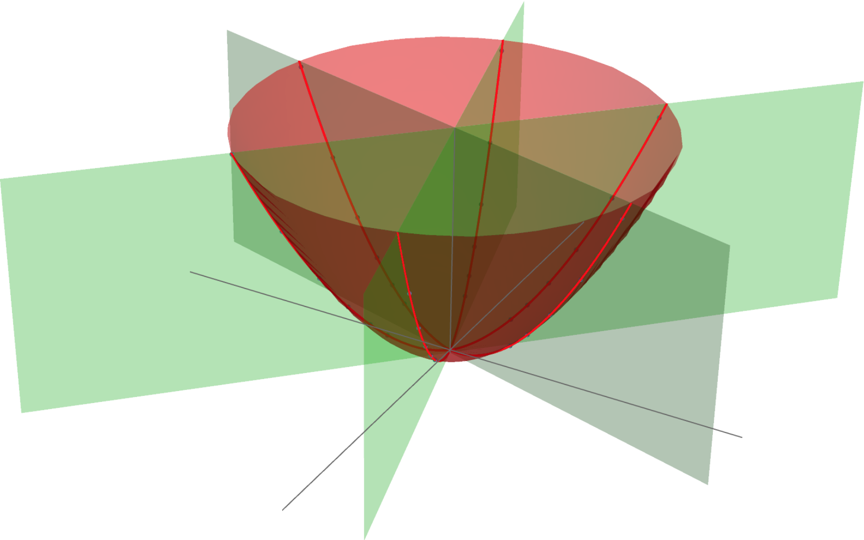

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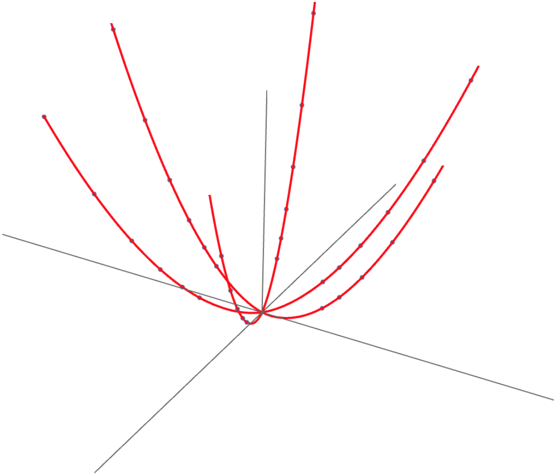

Main result

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There is an algorithm which computes, for any given affine program over $\overline{\mathbb{Q}}$, its strongest polynomial inductive invariant.

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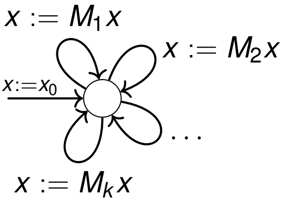
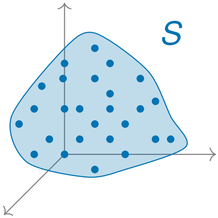
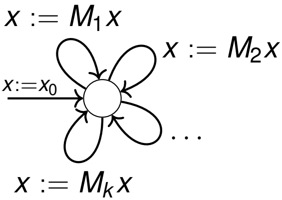
- Thus our algorithm computes all polynomial relations that always hold among program variables at each program location, in all possible executions of the program
- We represent this using a finite basis of polynomial equalities

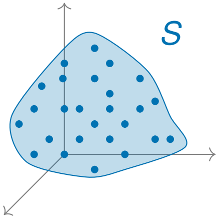
At the edge of decidability





At the edge of decidability



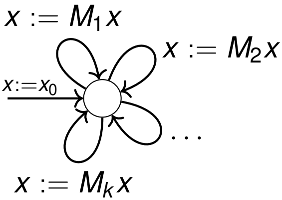


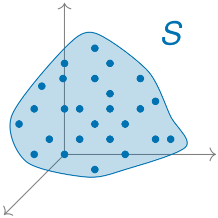
Theorem (Markov 1947[§])

There is a fixed set of 6×6 *integer matrices* $M_1, \dots, M_k$ *such that the reachability problem "*y *is reachable from* x_0*?" is undecidable.*

[§]Original theorems about semigroups, reformulated with affine programs.

At the edge of decidability





Theorem (Markov 1947§)

There is a fixed set of 6×6 *integer matrices* $M_1, \dots, M_k$ *such that the reachability problem "y is reachable from x_0?" is undecidable.*

Theorem (Paterson 1970*)

The mortality problem "0 is reachable from x_0 with $M_1, \dots, M_k$?" is undecidable for 3×3 *matrices.*

§Original theorems about semigroups, reformulated with affine programs.

Zariski closure of finitely generated groups

Our algorithm relies on this result:

Quantum automata and algebraic groups

Harm Derksen[a], Emmanuel Jeandel[b], Pascal Koiran[b,*]

[a]*Department of Mathematics, University of Michigan, Ann Arbor, MI 48109, United States*
[b]*Laboratoire de l'Informatique du Parallélisme, Ecole Normale Supérieure de Lyon, 69364, France*

Received 15 September 2003; accepted 1 November 2004

Theorem (Derksen, Jeandel and Koiran, 2004)

There is an algorithm which computes, for any given affine program over $\mathbb{Q}$ using only invertible transformations, its strongest polynomial inductive invariant.

Equivalently, compute the Zariski closure of a finitely generated groups of matrices.

From groups to semigroup

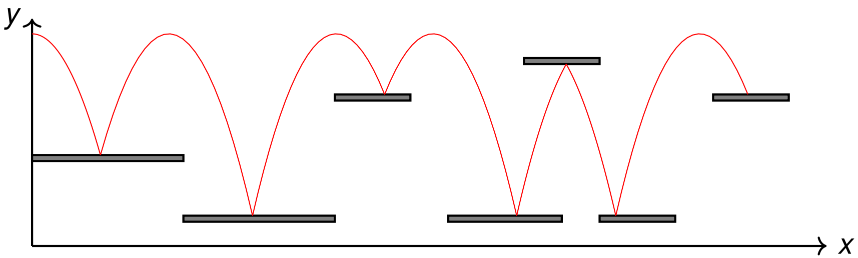
Theorem (Hrushovski, Ouaknine, P., Worrell, 2018)

There is an algorithm that computes the Zariski closure of any finitely semigroup of matrices (with algebraic coefficients), given its generators as inputs.

Corollary

Given an affine program, we can compute for each location the ideal of all polynomial relations that hold at that location.

Going hybrid: a bouncing ball



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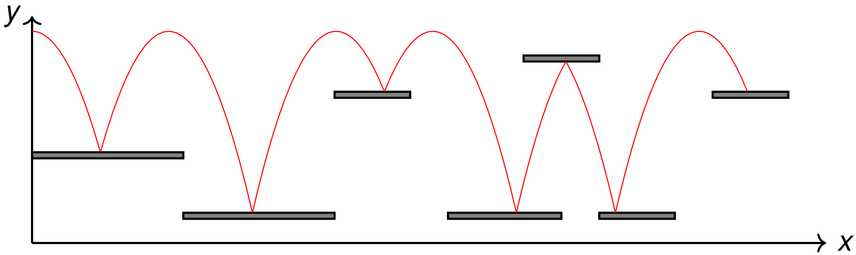
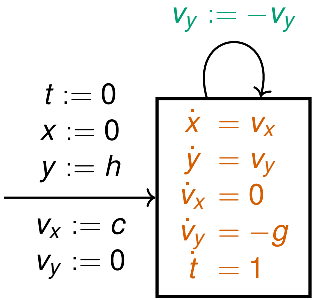
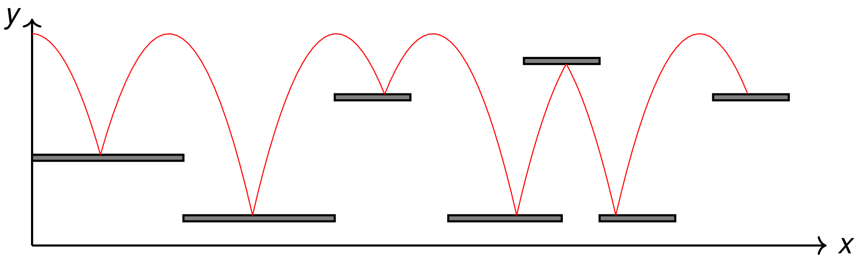
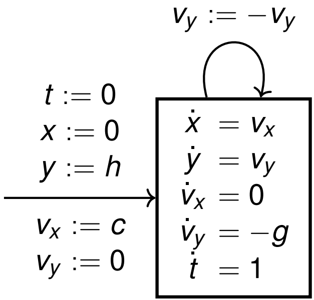




- affine program: collision
- $+$ linear differential equation: mechanics
- $=$ linear hybrid automaton

Going hybrid: a bouncing ball





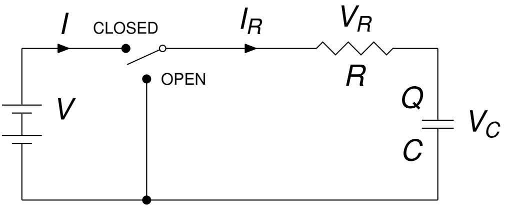
- affine program: collision
- \+ linear differential equation: mechanics
- = linear hybrid automaton

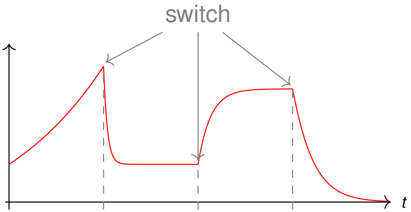
Invariants:

- $v_x = c$
- $x = tc$
- $v_y^2 + 2g(y - h) = 0$

recover conservation of energy!

Example: RC circuit





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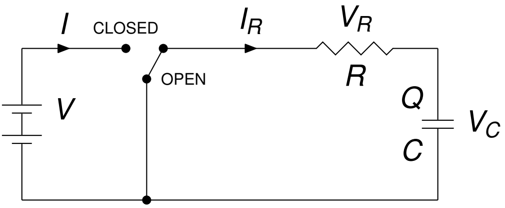
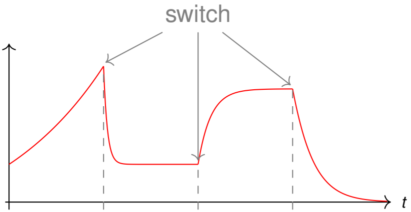
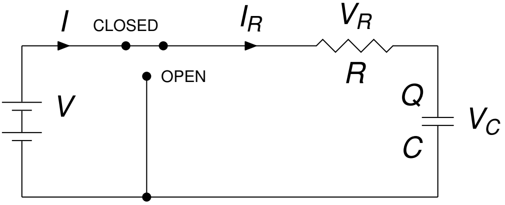


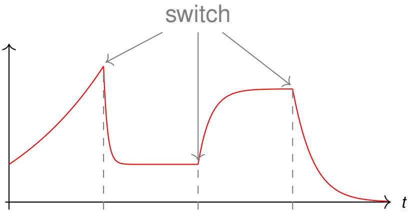


OPEN

$$\boxed{\begin{aligned} \dot{I} &= 0 \\ \dot{I}_R &= -\frac{1}{RC} I_R \\ \dot{V}_R &= -\frac{1}{C} I_R \\ \dot{Q} &= I_R \\ \dot{V}_C &= \frac{1}{C} I_R \end{aligned}}$$

Example: RC circuit





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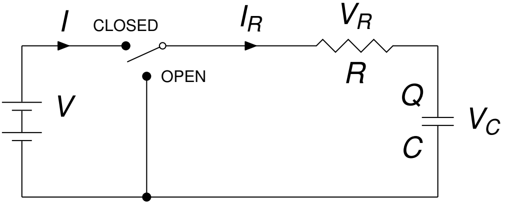

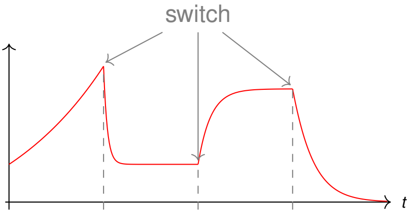

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OPEN → CLOSED:

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CLOSED → OPEN:

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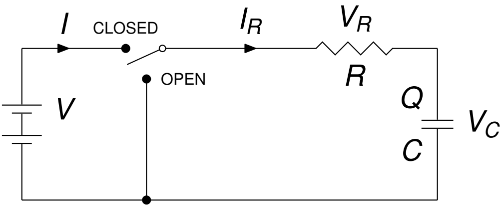

Invariants

OPEN	CLOSED
$Q = CV_C$	$Q = CV_C$
$V_R = RI_R$	$V_R = RI_R$
$I = 0$	$I = I_R$
$V_R = -V_C$	$V_R = V - V_C$

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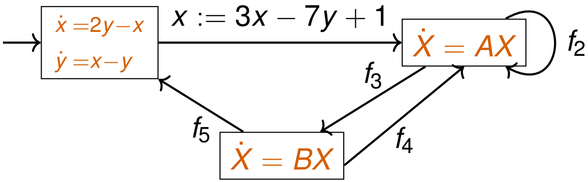
Linear Hybrid Automata

- Nondeterministic branching (no guards)
- All assignments are affine
- Linear differential equations in each location



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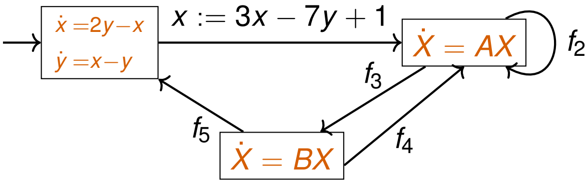

- More general than affine programs
- More general than linear differential equations

From affine programs to hybrid automata

Theorem (Majumdar, Ouaknine, P., Worrell, 2020)

There is an algorithm that computes, for any given guard-free linear hybrid automaton over $\overline{\mathbb{Q}}$, its strongest polynomial inductive invariant.

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For systems with purely continuous dynamics, *i.e.* no discrete transitions, called switching systems:

Theorem (Hrushovski, Ouaknine, P., Worrell, 2018)

There is no algorithm that computes the strongest algebraic inductive invariant for the class of switching systems with equality guards.

From hybrid automata to affine programs

Theorem (Majumdar, Ouaknine, P., Worrell, 2020)

There is an algorithm that computes, for any given ***guard-free linear hybrid automaton*** *over $\mathbb{Q}$, an* ***affine program*** *over $\mathbb{Q}$ that has the same polynomial inductive invariants.*

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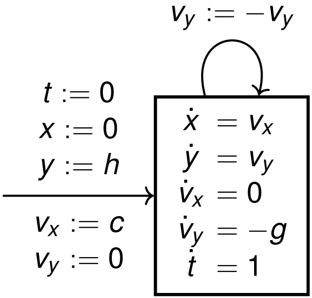

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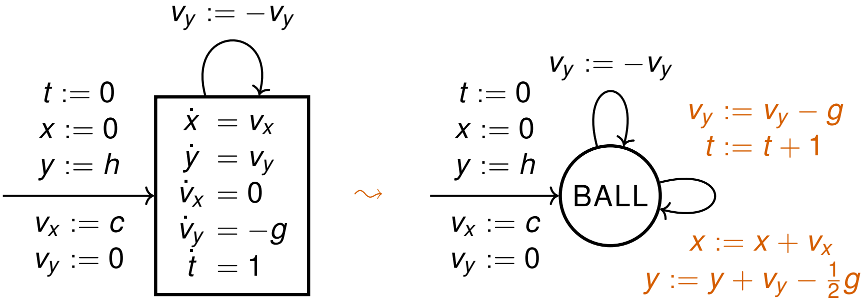

Linear Differential Equations

For $x(t) \in \mathbb{R}^n$ and A rational matrix, consider

$$\dot{x} = Ax$$

The solution is

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- strongest algebraic invariant = smallest algebraic set
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- obvious candidate $B = e^A$ is not algebraic
- "reverse-engineer" B algebraic to encode some multiplicative relations between the eigenvalues

Complexity of computing the Zariski closure

How expensive is it to compute this strongest invariant ?

$$\boxed{\text{linear hybrid}} \xrightarrow{\text{reduce}} \boxed{\text{semigroup}} \xrightarrow{\text{reduce}} \boxed{\text{group}}$$

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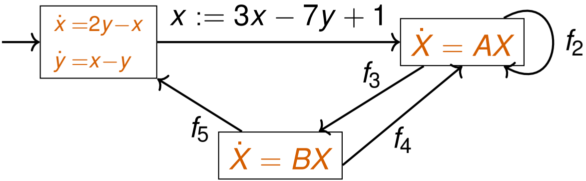
Theorem (Nosan, P., Schmitz, Shirmohammadi, Worrell, 2022)

Given a finite set S of invertible matrices of dimension n, the algebraic group $G := \overline{\langle S \rangle}$ can be defined with equations of degree at most septuply exponential in n.

Summary

- invariant = overapproximation of reachable states
- invariants allow verification of safety properties
- guard-free linear hybrid automata:
  - nondeterministic branching, no guards, affine assignments
  - linear differential equations



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